# Economics 203: Section 9

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## 1 Logistics

- Your final exam will be Thursday, March 20th, from 3:30 to 6:30 pm, in Meyer Forum (Room 124 in the Meyer Library). It will be closed book.
- I will hold a review session next week, as well as extra office hours. We will determine the time for the review session in class today. Please also let me know if there are any topics or problems you would like to see covered during the session.
- Course evaluations are now available on Axess. Please fill them out, as they really help me and Doug be better teachers.

## 2 Concepts

By this point in the class, we have seen all of the major solution concepts. Today in lecture, Doug will introduce two refinements on these concepts for games of incomplete information. I will just give the intuition, since you have not yet seen the formal definitions:

**Equilibrium Dominance:** (Also called the Ituitive Criterion.) If an off-path action is observed, it must be from the high type.

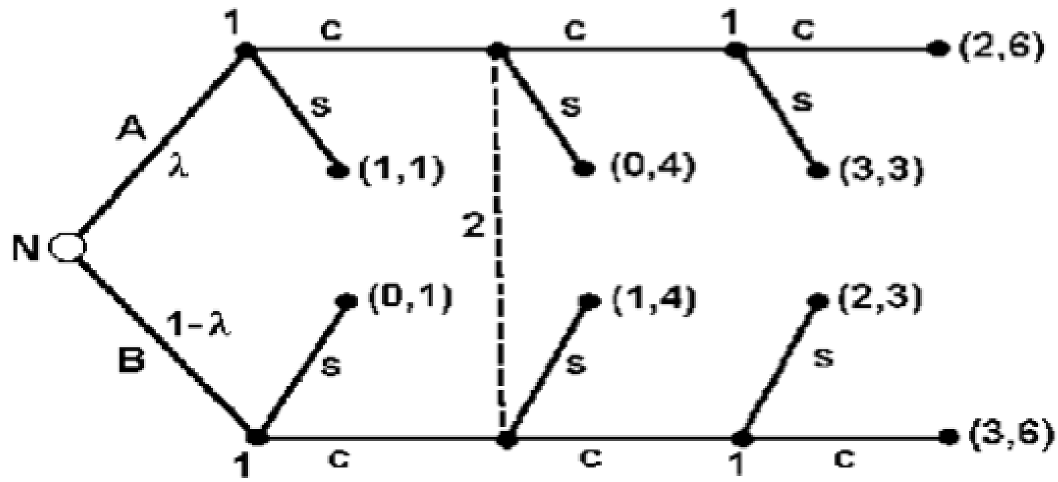
**D1:** If an off-path action is observed, it must be made by the type that stands to benefit more from taking that action.

The primary application discussed in this section is the Spence signaling model. Because this model is so pervasive in the field (and because it appears very often in exams), you should be able to easily reproduce every step of the derivation of the PBE of this game. Because you've not yet covered the model fully in lecture, we won't talk about it today. I will make sure to go over it during the review session.

## 3 Problems

**Problem 1.** *Problem Bank #100.*

Consider the following game of incomplete information:




The top half of this is an abbreviated version of the centipede game. The bottom half is similar, except that player 1 believes that continuation is always better for him. One might interpret this game as follows: Player 2 is playing the centipede game against an opponent, but is not certain that his opponent understands the game. The opponent is aware of player 2's uncertainty in this regard.

(a) Solve for the subgame perfect equilibria of this game.

(b) Solve for the sequential equilibria of this game. Does behavior unravel, as in the simple centipede game with complete information? What happens as $\lambda$ approaches 1? What does this tell you about how people might play the centipede game when they are unsure about each others' motives?

**Problem 2.** *Problem Bank #110.*

The President is deciding between two alternatives: enact a new educational program ($E$) or not enact and stay with the status quo ($N$). The status quo will give the president a (commonly known) payoff of 0. The payoff from the new education program depends on the state of the world. With probability $\frac{1}{4}$, the new program will be good ($G$) and give the President a payoff of 1. With probability $\frac{3}{4}$, it will be bad ($B$) and provide a payoff of -1.

Hoping to obtain sage advice, the President hires an Economist. He is uncertain, however, of the Economist's competance. The Economist is knowledgeable with probability $\frac{1}{4}$, and ignorant with probability $\frac{3}{4}$. If the Economist is knowledgeable ($K$), he is aware of the state of the world. If the Economist is ignorant ($I$), he does not know the state of the world (and therefore believes that the program will be good with probability $\frac{1}{4}$). The Economist knows his own type.

The game proceeds as follows:

Stage 1: Nature chooses the state of the world ($G$ or $B$) and the type of Economist ($K$ or $I$). The knowledgable Economist observes the state of the world.

Stage 2: The Economist advises the President about the state of the world. Specifically, he sends one of two messages, either $g$ (for good) or $b$ (for bad). The Economist cannot profess ignorance, and is constrained to send one of those two messages.

Stage 3: The President, viewing the advice, chooses either $E$ or $N$.

After viewing his payoff (either $-1$, 0, or $+1$), the President, who is a good Bayesian, infers that the Economist is knowledgable with probability $\mu$. The Economist's payoff is simply $\mu$ (he cares about his future reputation). All players are risk-neutral. Throughout, apply the PBE equilibrium concept.

(a) Identify each player's strategy set. (You may treat the Economist as either one player who learns his type, or as two players, one for each type.)

(b) Now you will explore the possibility that there is an equilibrium in which the knowledgeable Economist gives informative advice based on the state of the world ($g$ when $G$ and $b$ when $B$), and the ignorant Economist always sends the message $b$.

  (i) In such an equilibrium, what would be the probability that the President receives the message $g$? What would be his belief about the state of the world given he receives $g$? What will the President do when he receives the message $g$?

  (ii) In such an equilibrium, what would be the probability that the President receives the message $b$? What would be the probability that he receives the message $b$ and that the state of the world is $G$? What is his belief about the state of the world given he receives the message $b$? What will the President do when he receives the message $b$?

  (iii) In such an equilibrium, what would be the President's Bayesian beliefs about the probability that the Economist is knowledgeable given each combination of message and outcome? (For nodes off-the-equilibrium path, you should report the beliefs as a range.)

  (iv) Is there an equilibrium of the form we have been examining? If so, characterize it completely. If not, explain why not. (Remember that the President's final beliefs about the Economist's type determine the Economist's payoff.)

(c) How do your answers to part (b) change if the probability of the Economist being knowledgable is $\frac{1}{2}$? Interpret your results.

# 4 Solutions

*Solution 1.*

(a) First, we consider the cases when $\lambda = 1$ and $\lambda = 0$. For $\lambda = 1$, I will model the game as consisting only of the top branch. Then backwards induction gives us the following strategy profile: 1 plays $ss$ and 2 plays $s$. Thus the game ends immediately, as in the standard centipede game. For $\lambda = 0$, I consider only the lower branch of the game. We find that 1 plays $cc$ and 2 plays $c$. In this case, cooperation is sustained through the end of the game.

Next, consider the general case when $\lambda \in (0, 1)$. In general, 1's strategies will indicate her actions at four information sets. Note that we can use backwards induction on the last decision nodes of the top and bottom branches. On the top, 1 will choose $s$, while on the bottom, he will choose $c$. This is as far as backwards induction will take us, so now we can look for NE of the "reduced" game where 1's final decision is as given by backwards induction.

Note that in this reduced game, player 2 has just two pure strategies, since she has two actions at just one information set. However, player 1 has two information sets and two actions, so he has 4 pure strategies. Thus we can create the normal form of the reduced game:

| | $c$ | $s$ |
|---|---|---|
| $cc$ | $(3, 6-3\lambda)$ | $(1-\lambda, 4)$ |
| $cs$ | $(3\lambda, 2\lambda+1)$ | $(0, 3\lambda+1)$ |
| $sc$ | $(3-2\lambda, 6-5\lambda)$ | $(1, 4-3\lambda)$ |
| $ss$ | $(\lambda, 1)$ | $(\lambda, 1)$ |

We can see that $cs$ is dominated by $cc$ and $ss$ is dominated by $sc$. (The assumptions $\lambda \in (0, 1)$ is important here.) We can eliminate these strategies and consider the reduced normal form here:

| | $c$ | $s$ |
|---|---|---|
| $cc$ | $(3, 6-3\lambda)$ | $(1-\lambda, 4)$ |
| $sc$ | $(3-2\lambda, 6-5\lambda)$ | $(1, 4-3\lambda)$ |

Next, note that if 2 is playing $c$ then 1's best response is always $cc$, while if 2 is playing $s$ then 1's best response is always $sc$. If 1 is playing $sc$ then

2's best response is always $c$. However, if 1 is playing $cc$ then 2's best response depends on $\lambda$. In particular, $c$ will be a best response to $cc$ if $6 - 3\lambda \geq 4$, or $\lambda \leq \frac{2}{3}$. This suggests we break the analysis up by cases.

- $0 < \lambda < \frac{2}{3}$. We can see that $s$ is dominated by $c$ in the 2-by-2 normal form above, so the unique NE of the reduced game is $(cc, c)$. Thus the unique SPNE of the full game is $(ccsc, c)$ in this case.

- $\lambda = \frac{2}{3}$. The normal form is then

| | $c$ | $s$ |
|---|---|---|
| $cc$ | $(3, 4)$ | $(\frac{1}{3}, 4)$ |
| $sc$ | $(1\frac{2}{3}, 2\frac{2}{3})$ | $(1, 2)$ |

  We see immediately that in a NE of this normal form, 1 cannot put any weight on $sc$. If he did, then 2's best response would be to play $c$, and then 1's best response to that would be to play $cc$, a contradiction. Thus in any NE, 1 plays only $cc$. In that case, 2 is indifferent between her two strategies. As long as 2 mixes with probability $p \leq \frac{2}{3}$ on $c$, 1 will not have an incentive to deviate. Thus the SPNE of the full game in this case are given by the set $\{(ccsc, pc + (1-p)s) : p \leq \frac{2}{3}\}$.

- $\frac{2}{3} < \lambda < 1$. In this case it is easy to see that there are no PSNE in the reduced game. Furthermore, since all best responses are unique, there will be no pure-mixed NE. This leaves only a full mixed NE. Suppose that 2 mixes with probability $q$ on $c$. Then 1's indifference between $cc$ and $cs$ gives us that $q = \frac{1}{3}$. (Note that this does not depend on $\lambda$!). Suppose that 1 mixes with probability $p$ on $cc$. Then 2's indifference between her strategies will give us that $p = \frac{2-2\lambda}{\lambda}$. (Note that $p \in (0, 1)$ since $\lambda \in (\frac{2}{3}, 1)$.) Thus the SPNE of the full game is $\left(\frac{2-2\lambda}{\lambda} ccsc + \frac{3\lambda-2}{\lambda} scsc, \frac{1}{3}c + \frac{2}{3}s\right)$ in this case.

(b) For the degenerate cases ($\lambda = 1$ and $\lambda = 0$): The SPNE found as also SE, since in these cases there are no non-singleton information sets. (Again, I am modeling the game in these degenerate cases as consisting only of a single branch.)

For the case $\lambda \in (0, 1)$: All non-singleton information sets (namely, 2's information set) are reached with positive probability every SPNE above. Thus 2's beliefs are given automatically by Bayes' rule and are guaranteed to be consistent. Let $\mu$ be 2's belief that he is at the upper node of his information set. Let's check consistency case-by-case:

- $0 < \lambda < \frac{2}{3}$: The candidate strategy profile, $(ccsc, c)$ tells us immediately that we must have $\mu = \lambda$, since 1 plays $c$ after $A$ and after $B$.

- $\lambda = \frac{2}{3}$: By the same logic as the previous case, we have $\mu = \lambda$.

- $\frac{2}{3} < \lambda < 1$: 1's mixed strategy has her always playing $c$ after $B$, and playing $c$ after $A$ with probability $\frac{2-2\lambda}{\lambda}$. Thus we must have $\mu = \frac{\lambda\frac{2-2\lambda}{\lambda}}{\lambda\frac{2-2\lambda}{\lambda}+1-\lambda} = \frac{2}{3}$.

Next, we note that all information sets are reached with positive probability on the equilibrium path, sequential rationality is guaranteed by the Nash conditions we checked in the previous part. Thus in each of the 3 cases above, the SE are the SPNE strategies plus the beliefs we just calculated.

Note that as $\lambda \to 1$, the strategies converge to $(scsc, \frac{1}{3}c + \frac{2}{3}s)$. That is, 1's strategy has him playing stop on the top branch – as in the case when $\lambda = 1$ – nearly all the time. But because of the infinitesimal chance that 1 is type $B$ (that is, we end up on the bottom branch), 2 plays stop only $\frac{2}{3}$ of the time.

*Solution 2.*

(a) I will treat the economist as a player with 3 information sets: (1) he is of the knowledgeable type and the state of the world is good, (2) he is knowledgeable and the state is bad, and (3) he is ignorant. Then we have the strategy set for the economist as $S_E = \{g, b\} \times \{g, b\} \times \{g, b\}$. The president's decision depends only on the signal she receives, so she has two actions at two information sets. Her strategy set is $S_P = \{E, N\} \times \{E, N\}$.

(b) (i) The president receives a good signal only if the economist is type $K$ and the state of the world is good. Thus $P(g) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$. Given that she receives a good signal, the president can conclude it came from a type $K$ economist and is accurate. Thus $P(G|g) = 1$. The president will play $E$ in this case, for a payoff of 1.

(ii) The president receives a bad signal if the economist is type $K$ and the state of the world is bad, or if the economist is of type $I$. Thus $P(g) = \frac{1}{4} \cdot \frac{3}{4} + \frac{3}{4} = \frac{15}{16}$. Given that she receives a bad signal, the president knows the state of the world is good with probability $\frac{3}{4} \cdot \frac{1}{4}$. Thus $P(G|b) = \frac{1}{5}$. The president will play $N$ in this case, for a payoff of 0, since playing E will give a negative expected payoff.

(iii) One the path, there are only two possibilities: If the program is successful, then the president must have seen signal $g$, which is only sent by the $K$ type economist. Thus $\mu = 1$. If the program is not enacted, then the president must have seen signal $b$, in which case $\mu = P(E = K|b) = \frac{1}{5}$. For the any other outcome, we can have any $\mu \in [0, 1]$.

(iv) We show that there is NOT a PBE of the form $(gbb, EN)$ for any beliefs of the president. To see why, note that if the type $I$ economist plays $b$, the president will play $N$, and the payoffs to the economist will be $\frac{1}{5}$. But if the economist plays $g$, the president will play $E$. The program will be successful with probability $\frac{1}{4}$, giving the economist payoff of 1. But the program will fail with probability $\frac{3}{4}$, in which case the economist's payoffs will be $\mu$. Note that for any $\mu$, the economist prefers to play $g$. Hence we cannot have a PBE, since sequential rationality is violated.

(c) (i) The president receives a good signal only if the economist is type $K$ and the state of the world is good. Thus $P(g) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$. Given that she receives a good signal, the president can conclude it came from a type $K$ economist and is accurate. Thus $P(G|g) = 1$. The president will play $E$ in this case, for a payoff of 1.

(ii) The president receives a bad signal if the economist is type $K$ and the state of the world is bad, or if the economist is of type $I$. Thus $P(g) = \frac{1}{4} \cdot \frac{1}{2} + \frac{3}{4} = \frac{7}{8}$. Given that she receives a bad signal, the president knows the state of the world is good with probability $\frac{1}{2} \cdot \frac{1}{4}$. Thus $P(G|b) = \frac{1}{7}$. The president will play $N$ in this case, for a payoff of 0, since playing E will give a negative expected payoff.

(iii) One the path, there are only two possibilities: If the program is successful, then the president must have seen signal $g$, which is only sent by the $K$ type economist. Thus $\mu = 1$. If the program is not enacted, then the president must have seen signal $b$, in which case $\mu = P(E = K|b) = \frac{3}{7}$. For all other possibilities, we can have any $\mu \in [0, 1]$.

(iv) We confirm that there is a PBE of the form $(gbb, EN)$ for some beliefs of the president.

First, the president is sequentially rational by the arguments above.

Next, the $K$ type economist is clearly sequentially rational for playing $g$ when the state of the world is good, as this gets him payoff 1. If the state of the world is bad, he gets payoff $\frac{3}{7}$ for playing $b$ and payoff $\mu$ for playing $g$. Thus we need $\mu \leq \frac{3}{7}$ for $b$ to be sequentially rational here.

Finally, note that if the type $I$ economist plays $b$, the president will play $N$, and the payoffs to the economist will be $\frac{3}{7}$. But if the economist plays $g$, the president will play $E$. The program will be successful with probability $\frac{1}{4}$, giving the economist payoff of 1. But the program will fail with probability $\frac{3}{4}$, in which case the economist's payoffs will be $\mu$. Thus if $\frac{3}{4}\mu + \frac{1}{4} \leq \frac{3}{7}$, or $\mu \leq \frac{5}{21}$, the $I$ type economist is sequentially rational in playing $b$.

Hence we have a PBE $((gbb, EN), \mu)$ for $\mu \leq \frac{5}{21}$ (and all other beliefs as given in previous parts).

The interpretation is that as knowledgeable economists become more rare, ignorant economist will have an incentive to try to emulate them, since the chance that they are caught is very low.