Economics 203: Section 6

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1 Logistics

Problem Sets 4 and 5 have been graded and are available to pick up in section and lecture. I have also released solutions through problem 58 on Coursework.

2 Concepts

2.1 Weak Perfect Bayesian Equilibrium

We have seen that SPNE can make strange predictions because it does not force players to make reasonable choices at non-singleton information sets. But to talk about players responding rationally at information sets, we need a formal way to talk about their beliefs about which node they are at in an an information set.

Definition 1. Let X be the set of decision nodes (so $t \in X$). Let H be the set of information sets for a particular information partition. Let $h(t)$ be the info set containing node $t \in X$, and $\varphi(h)$ be the player making the decision at info set $h \in X$. Then a **system of beliefs** is a mapping $\mu : X \to [0, 1]$ such that $\sum_{t \in h} \mu(t) = 1$ for all $h \in H$.

So, beliefs tell a decision maker the probability with which they are at a particular node in a given information set. Note that the beliefs within an info set add to 1, since $\mu(t_1)$ for $t_1 \in h_1$ is the probability that the game has reached node t_1 *conditional* on the fact that it has reached info set h_1.

Definition 2. A behavior strategy profile δ is **sequentially rational** given a system of beliefs μ iff for all h the actions of $\varphi(h)$ at $h \cup [\cup_{t \in h} S(t)]$ are optimal starting from h given an initial probability over h governed by μ and given that other players adhere to δ.

Sequential rationality is a way of generalizing the intuition of subgame perfection to every information sets, with beliefs μ determining the probabilities that the subgame is starting at a particular node within each info set h.

Definition 3. The pair (δ^*, μ^*) is a **weak perfect Bayesian equilibrium (WPBE)** iff

(i) the behavior strategy δ^* is sequentially rational given μ^*, and

(ii) where possible, μ^* is computed from δ^* using Bayes' rule. That is, for h s.t. $P(h|\delta^*) > 0$, we have

$$\mu^*(t) = \frac{P(t|\delta^*)}{P(h|\delta^*)}.$$

Note, however, that out of equilibrium beliefs (those where $P(h|\delta^*) = 0$) are *not* restricted by Bayes' rule. This extra flexibility can lead to WPBE that do not seem reasonable, essentially because they include a non-credible threat off the equilibrium path. To eliminate this problem, we can combine subgame perfection with WPBE:

Definition 4. A **perfect Bayesian equilibrium** is a WPBE that is also a WPBE in every proper subgame.

Note that if a game has no proper subgames, WPBE = PBE.

2.2 Sequential Equilibrium

With WPBE and PBE equilibrium, we restrict that players best-respond to *some* beliefs at every info set. But since these beliefs are unrestricted off the equilibrium path, we still can get strange equilibria. However, we can require beliefs be consistent in the sense that they are formed from strategy profiles that reach every info set with positive probability, but that are not too different from the equilibrium profile.

Definition 5. A behavior strategy profile δ is **strictly mixed** iff every action at every info set is selected with positive probability.

Note that if δ is strictly mixed, all beliefs μ can be calculated from Bayes' rule. Call this relation $\mu = M(\delta)$.

Definition 6. The pair (δ, μ) is **consistent** iff there exists a sequence $\delta^n \to \delta$ of strictly mixed behavior strategy profiles such that $\mu^n \to \mu$, where $\mu^n = M(\delta^n)$.

Definition 7. A pair (δ^*, μ^*) is a **sequential equilibrium (SE)** if it is consistent and sequentially rational.

2.3 Finding WBPE, PBE, and SE

The most important thing to remember is that each of these equilibrium concepts require that you give both strategies and beliefs for each player. The trick is then determining which combinations of strategies and beliefs are valid equilibria.

It can also be helpful to remember the following relationship between solution concepts:

$$\text{SE} \subseteq \text{PBE} \subseteq \left\{ \begin{array}{c} \text{WPBE} \\ \text{SPNE} \end{array} \right\} \subseteq \text{NE}$$

This suggests an order of operations when looking for equilibria, as we will see in the checklists below.

Here is a checklist we can follow to find WPBE:

1. If all information sets are singletons, note that SE = PBE = WPBE = SPNE, so just find SPNE and we are done.
2. Write down the game in normal form and find the pure-strategy NE. You will often be asked to find pure-strategy Bayesian or sequential equilibria, and these will naturally be subsets of the pure-strategy NE. It's also a good idea to look out for dominated strategies, as these will not be played in any equilibrium.
3. Propose a WPBE. This will consist of NE strategies plus beliefs for any player that has a non-singleton info set.
4. For a proposed WPBE, check that we have sequential rationality and that Bayes' rule is followed where applicable. If these two conditions lead to a contradiction, the proposed equilibrium is not a WPBE. Otherwise, we have found a (family of) WPBE.
    (a) In general, start with checking Bayes' rule (where applicable) for each player.
    (b) Finally, check sequential rationality for each player. Note that a player's sequential rationality condition will restrict that player's beliefs, or actions by either player, but not that player's opponent's beliefs.

Here is a checklist to find PBE:

1. Start with the WPBE found by the above.
2. Note all the subgames of the game. If there are no proper subgames, note that PBE = WBPE, so we are done.

3. Apply the checklist for WPBE for each subgame. Note that any proposed equilibrium must include strategies from a SPNE.

Lastly, here is checklist for finding SE:

1. Start with the PBE found by above.
2. Note that you simply need to check for consistency, since the PBE are already sequentially rational.
   (a) Note that if a proposed equilibrium reaches every info set with positive probability, then it is necessarily consistent.[1]
   (b) If every info set is *not* reached in a proposed equilibrium, then we will have to show consistency directly, or prove that consistency is not possible for this equilibrium. This is the only part of the process that is not mechanical, and may require some intuition or guesswork. In general, to show some (δ, μ) is a SE, do the following:
      - Pick a sequence $\delta^n \to \delta$. A common trick is to parameterize the sequence by something like ϵ^n, where $\epsilon < 1$ is a constant and $n \to \infty$. The exact parameterization will depend on what you want the sequence of beliefs and the sequence of strategies to converge to.
      - Calculate μ^n using Bayes' Rule.
      - Show that $\mu^n \to \mu$.

3 Problems

Problem 1. *Problem Bank # 58.*

Consider the following game played by two parties, A and B. First, "nature" chooses either C or D. C is chosen with probability 0.7, and D is chosen with probability 0.3. Second, party A chooses either E or F. Party A does not observe nature's choice when it makes this choice. Next, party B chooses either G or H. Prior to making this choice, party B observes the choice of party A; party B also observes nature's choice if A has chosen E, but does not observe nature's choice if A has chosen F. Payoffs are determined as follows: A and B always receive the same payoff; the payoff is 0 if A chooses E and B chooses G, regardless of nature's choice; the payoff is 5 if A chooses E and B chooses H, regardless of nature's choice; the payoff is 0 if nature chooses C, A chooses F, and B chooses G; the payoff is 10 if nature chooses C, A chooses F, and B

[1] In this is not clear to you, we'll see this feature in the problems shortly.

chooses H; the payoff is 10 if nature chooses D, A chooses F, and B chooses G; and the payoff is 0 if nature chooses D, A chooses F, and B chooses H.

(a) Draw the extensive form of this game.

(b) Identify the strategy set for each player. Write the normal form of the game.

(c) Identify all pure strategy Nash equilibria for this game.

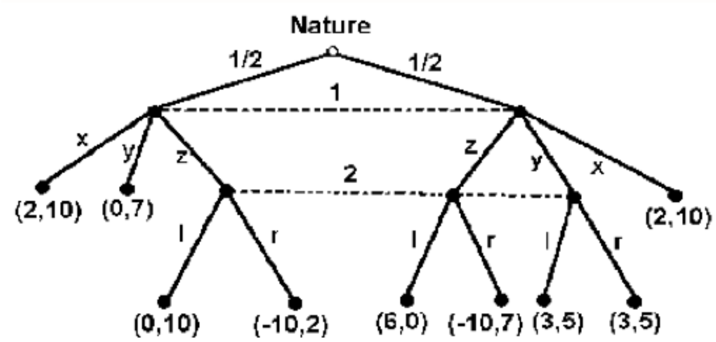
(d) Identify all pure strategy subgame perfect Nash equilibria for this game.

(e) Identify all pure strategy Bayesian perfect equilibria for this game.

(f) Identify all pure strategy sequential equilibria for this game.

Problem 2. *Problem Bank # 57.*

Consider the following two player game in extensive form:



(a) Characterize the set of Perfect Bayesian Equilibria.

(b) Characterize the set of Sequential Equilibria.

4 Solutions

Solution 1.

(a) The extensive form is given in Figure 1.

(b) For player A, the strategy set is $\{E, F\}$, since she has just one information set. For player B, the strategy set is $\{G, H\} \times \{G, H\} \times \{G, H\}$, since he can choose to play G or H are each of his three information sets. The normal form is given below:

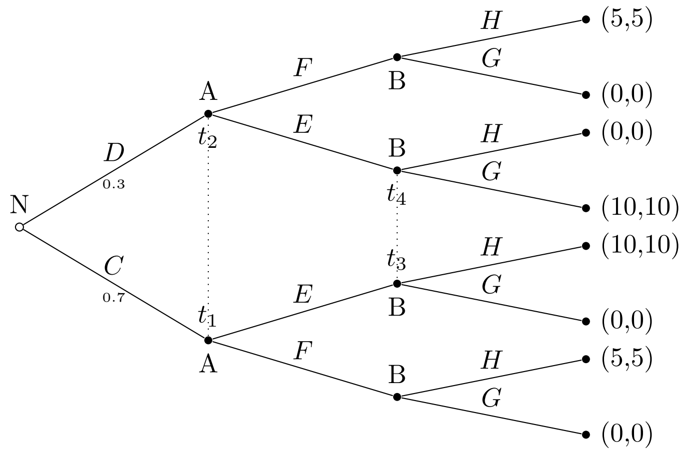


Figure 1: The extensive form for the game in Problem 1.

	GGG	GGH	GHG	GHH	HGG	HGH	HHG	HHH
E	$(0,0)$	$(1.5,1.5)$	$(0,0)$	$(1.5,1.5)$	$(3.5,3.5)$	$(5,5)$	$(3.5,3.5)$	$(5,5)$
F	$(3,3)$	$(3,3)$	$(7,7)$	$(7,7)$	$(3,3)$	$(3,3)$	$(7,7)$	$(7,7)$

(c) From the normal form we can immediately see that there are five PSNE: (E, HGH), (F, GHG), (F, GHH), (F, HHG), (F, HHH).

(d) There are only two proper subgames: after nature plays C and A plays E, and after nature plays D and A plays E. In both of these subgames, B must choose H. Thus the only pure strategy SPNE are (E, HGH) and (F, HHH).

(e) Note that sequential rationality implies that B plays H after A plays E. Thus the only possible pure strategy PBE are the two SPNE we found in the previous part. For notational purposes, let $\lambda = \mu(t_1)$ parameterize A's beliefs and $\gamma = \mu(t_3)$ parameterize B's beliefs.

- Consider (E, HGH). Note that we must have $\lambda = 0.7$, of course, for A's beliefs. Since B's non-singleton information set is not reached, B's beliefs γ are unrestricted by Bayes' rule. However, for HGH to be sequentially rational we need to have $10\gamma \leq 10(1-\gamma)$, or $\gamma \leq 1/2$. And given that B is playing HGH, E is clearly sequentially rational for A (since E gives a payoff of 5 but F gives a payoff of 3). Thus $(\delta = (E, HGH), \mu = (\lambda, \gamma))$ is a PBE for $\lambda = 0.7$ and $\gamma \leq 1/2$.

- Consider (F, HHH). As above, we must have $\lambda = 0.7$. Since B's information set is reach with positive probability, Bayes' rule gives us $\gamma = \frac{0.7(1)}{0.7(1)+0.3(1)} = 0.7$ as well. With beliefs now fixed, note that

HHH is sequentially rational for B, since $\gamma \geq 1/2$. And note that assuming B is playing HHH, A gets 5 for playing E but 7 for playing F, so F is sequentially rational. Thus $(\delta = (F, HHH), \mu = (\lambda, \gamma))$ is a PBE for $\lambda = 0.7$ and $\gamma = 0.7$.

(f) Since we have already identified the two PBE, we simply need to check them for consistency.

- Consider $\delta = (E, HGH)$. Let δ^n be an arbitrary strictly mixed strategy profile such that $\delta^n \to \delta$, and in particular, let δ_1^n be the probability A plays E. Then by Bayes' rule we must have $\gamma_n = \frac{0.7(1-\delta_1^n)}{0.7(1-\delta_1^n)+0.3(1-\delta_1^n)} = 0.7$. Thus consistency requires that we have $\gamma^n \to \gamma = 0.7$. But note that if $\gamma = 0.7$ then for sequential rationality we must have B play H at his information set. Thus we have a contradiction, so this cannot be a SE.
- Consider $\delta = (F, HHH)$. By the logic above, we must have $\gamma = 0.7$ for consistency, and by the previous part we already know that sequential rationality is satisfied. Thus $(\delta = (F, HHH), \mu = (\lambda, \gamma))$ is a SE for $\lambda = 0.7$ and $\gamma = 0.7$.

Solution 2.

(a) Note that the game has no proper subgames, so WPBE = PBE. Also, we can see from the normal form that y is dominated for 1:

	l	r
x	$(2, 10)$	$(2, 10)$
y	$(1.5, 6)$	$(1.5, 6)$
z	$(3, 5)$	$(-10, 4.5)$

So 1 will not play y in any equilibrium. This means we can consider just two cases: one where 1 plays x only, and one where 1 plays z with positive probability. Note that I chose to divide the cases this way because in the first 2's info set is never reached, while in the second it is reached with positive probability.[2]

For notational purposes, let λ be 1's belief that nature has moved left. Let γ_1 be 1's beliefs that nature has moved left and 1 has chosen z (call this node t_3), and γ_2 be his beliefs that nature has moved right and 1 has played z (call this t_4). Note that in any equilibrium we must have $\lambda = 1/2$. Let h_1 be 1's info set and h_2 be 2's info set.

- Suppose 1 plays x only in a PBE. Then let's say 2's strategy is given by $(\delta_2, 1-\delta_2)$; that is, he puts probability δ_2 on playing l if his info

[2] Yes, I am deviating a bit from my own checklist. I'm doing this to save a bit of math, and because the fact that the game ends after x gives us an easy way to partition the cases. However, you could also proceed as follows: Find all the NE. You'll find two PSNE, plus one NE where 1 plays x and 2 mixes between l and r. Then for each of these potential PBE, find beliefs that satisfy sequentially rationality and, if applicable, Bayes' rule.

set is reached. Note that Bayes' rule does not have any bite for player 2 in this case. Next, let's check sequentially rationality, starting with player 1. If she pays x, she gets payoff of 2. If she plays z, she gets payoff

$$\frac{1}{2}\delta_2 0 + \frac{1}{2}\delta_2 6 + \frac{1}{2}(1-\delta_2)(-10) + \frac{1}{2}(1-\delta_2)(-10) = 13\delta_2 - 10.$$

Thus for 1's sequential rationality we must have $13\delta_2 - 10 \leq 2$, or $\delta_2 \leq {}^{12}/_{13}$.

Next, we need to check 2's sequential rationality. First, suppose $\delta_2 = 0$.[3] Then player 2's payoff from r must be weekly greater than his payoffs from l:

$$2\gamma_1 + 7\gamma_2 + 5(1-\gamma_1-\gamma_2) \geq 10\gamma_1 + 0\gamma_2 + 5(1-\gamma_1-\gamma_2),$$

which gives $8\gamma_1 \leq 7\gamma_2$. Thus

$$(\delta = ((1,0,0),(0,1)), \mu = ((1/2, 1/2), (\gamma_1, \gamma_2, 1-\gamma_1-\gamma_2)))$$

is a PBE when $8\gamma_1 \leq 7\gamma_2$.

Next, suppose ${}^{12}/_{13} \geq \delta > 0$. Then 2's sequential rationality requires that he is indifferent between l and r given his beliefs. That is, $8\gamma_1 = 7\gamma_2$. Thus

$$(\delta = ((1,0,0),(\delta_2, 1-\delta_2)), \mu = ((1/2, 1/2), (\gamma_1, \gamma_2, 1-\gamma_1-\gamma_2)))$$

is a PBE when $8\gamma_1 = 7\gamma_2$ and ${}^{12}/_{13} \geq \delta > 0$.

It is important to note that in both of these sub-cases, player 2's beliefs might put positive probability on the possibility that player 1 has played y if 2's info set is reached, *even though* both players know that y is dominated.

- Suppose that 1 puts some weight on playing z. Then we can write her strategy as $(\delta_1, 0, 1-\delta_1)$, where δ_1 is the probability that she plays x. Note we can assume she does not put any weight on playing y. Now that 2's info set is reached, we can apply Bayes' rule:

$$\gamma_1 = P(t_3|h_2,\delta) = \frac{P(t_3|\delta)}{P(h_2|\delta)} = \frac{\frac{1}{2}(1-\delta_1)}{\frac{1}{2}(1-\delta_1)+\frac{1}{2}(1-\delta_1)+\frac{1}{2}0} = \frac{1}{2}.$$

We can show similarly that $\gamma_2 = \frac{1}{2}$ as well, and $\gamma_3 = 0$. Now that beliefs are nailed down, it is straightforward to check for sequential rationality. Given his beliefs, player 2 chooses to play l

[3] Why break up the subcases this way? Because I know that (x, r) is a NE.

with probability 1 for expected payoff 5, since r gives payoffs 4.5 in expectation. And given this strategy by 2, 1's sequential rationality requires that she play z with probability 1 for expected payoff of 3, which is greater than her payoff of 2 for playing x. Thus $(\delta = ((0,0,1),(1,0)), \mu = ((1/2, 1/2),(1/2, 1/2, 0)))$ is a PBE.

(b) Note that since we have already found the PBE, we simply need to check these for consistency to see if they are SE.

- Consider our PBE where 1 plays x for sure and 2 plays r for sure. In this case, to show consistency we need to find some strictly mixed δ^n such that $\mu^n = M(\delta^n) \to \mu$, where beliefs μ must be such that $7\gamma_1 \leq 8\gamma_2$. However, note also that by Baye's rule, $\gamma_1^n = \gamma_2^n$ for any n! Thus these two series must have the same limits, so that $\gamma_1 = \gamma_2$. Therefore, our two conditions on γ_1 and γ_2 require $\gamma_1 = \gamma_2 = 0$.

  To achieve these limits, we need to pick a sequence of strictly mixed strategies for that converge to $(1,0,0)$ for player 1. Let's say that $\delta_1^n = (\sigma_1^n, \sigma_2^n, 1-\sigma_1^n-\sigma_2^n)$, noting that we have $\sigma_1^n \to 1$ and $\sigma_2^n \to 0$. Then Bayes' rule says that

  $$\gamma_1^n = \frac{\frac{1}{2}(1-\sigma_1^n-\sigma_2^n)}{\frac{1}{2}(1-\sigma_1^n-\sigma_2^n)+\frac{1}{2}(1-\sigma_1^n-\sigma_2^n)+\frac{1}{2}\sigma_2^n}.$$

  Note that both the numerator and the denominator must go to 0 in the limit, but the fraction must go to 0 as well, so we need the numerator to shrink faster than the denominator. That is, we need $\sigma_1^n \to 1$ faster than $\sigma_2^n \to 0$. One possibility is $\sigma_1^n = 1-\varepsilon^{2n}-\varepsilon^n$ and $\sigma_2^n = \varepsilon^n$, where $\varepsilon > 0$ and $n \to \infty$. Then

  $$\gamma_1^n = \frac{\frac{1}{2}\varepsilon^{2n}}{\frac{1}{2}\varepsilon^{2n}+\frac{1}{2}\varepsilon^{2n}+\frac{1}{2}\varepsilon^n} = \frac{\varepsilon^n}{2\varepsilon^n+1}.$$

  Note that the numerator of the last expression goes to 0 in the limit while the denominator goes to 1, so the overall limit is 0 as desired.

  Lastly, we also technically need a strictly mixed strategy for 2, but this has no effects on beliefs, so any sequence that converges to $(0,1)$ works. Thus we have consistency, and so

  $$(\delta = ((1,0,0),(0,1)), \mu = ((1/2, 1/2),(0,0,1)))$$

  is a SE.

- Next, consider our PBE where 1 plays x for sure and 2 mixes between l and r. Note, however, that the above results still apply, since $7\gamma_1 = 8\gamma_2$ and $\gamma_1 = \gamma_2$ still imply $\gamma_1 = \gamma_2 = 0$! Thus we have that $(\delta = ((1,0,0),(\delta_2, 1-\delta_2)), \mu = ((1/2, 1/2),(0,0,1)))$ is a SE for $0 < \delta_2 \leq 12/13$.

- Lastly, consider our PBE when 1 plays z for sure. In this case, note that 2's info set is reached with positive probability, so consistency is already guaranteed. To see this more explicitly, note that by Bayes' rule,

  $$\gamma_1^n = \frac{\frac{1}{2}(1-\sigma_1^n-\sigma_2^n)}{\frac{1}{2}(1-\sigma_1^n-\sigma_2^n)+\frac{1}{2}(1-\sigma_1^n-\sigma_2^n)+\frac{1}{2}\sigma_2^n} \to \frac{1}{2},$$

  since $\sigma_1^n \to 0$ and $\sigma_2^n \to 0$, regardless of the particular sequence of strictly mixed strategies. Thus

  $$(\delta = ((0,0,1),(1,0)), \mu = ((1/2, 1/2), (1/2, 1/2, 0)))$$

  is a SE.