Economics 203: Section 3

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1 Logistics

1.1 Problem Set Graded

Problem set 1 has been graded and is available for pickup in section or lecture. If you got a check next to your name, you got credit for this problem set.

Some feedback from common mistakes and issues I saw while grading:

- Save yourself some writing when describing strategies:
  - Pick a useful shorthand for actions (eg. H for heads).
  - Actions don't need unique names (eg. hh is fine to stand in for "heads after heads, heads after tails").
  - It's OK to write down a strategy set with product notation (eg. $\{H,T\} \times \{H,T\} \times \{H,T\} \times \{H,T\}$ instead of the 16 strategies that implies).
- But don't cut out too much writing:
  - Write big and clear so I can read what you've put down.
  - Show your work and reasoning. (This will not affect your problem set grades, but you can get partial credit on exams, so best to practice that now.)
- If writing out a normal form requires taking an expectation over nature, do this step! Don't just describe what math is required.
- Read problems carefully. Several of you wrote down extensive forms incorrectly because you missed an action for a player.
- Don't iteratively delete when asked about weakly dominated strategies.
- In problem 3, you found the formula for the size of a player's strategy set. Check against this formula when writing out the normal form!

1.2 Problem Bank Solutions Released

I have uploaded the solutions through problem set 1. I will continue to add segments of the solutions each week throughout the quarter.

2 Concepts

2.1 Pure Strategy Nash Equilibrium

Last week's lectures focused mostly on applications of pure strategy Nash equilibrium. As a reminder, here is the definition:

Definition 1. A strategy profile $s^* = (s_1^*, \ldots, s_I^*)$ is a **pure strategy Nash equilibrium (PSNE)** iff

$$g_i(s_i^*, s_{-i}^*) \geq g_i(s_i, s_{-i}^*)$$

for all $s_i \in S_i$ and for all $i \in I$.

2.1.1 Finding PSNE

As we talked about last week, it is often useful to write down the **best-response correspondence** of a player i given the other players' strategies,

$$BR_i(s_{-i}) = \arg\max_{s_i \in S_i} g_i(s_i, s_{-i}).$$

We can then write an overall best-response correspondence as

$$BR(s) = (BR_1(s_{-1}), \ldots, BR_I(s_{-I})).$$

A PSNE s^* is a fixed point of this correspondence: $s^* = BR(s^*)$. Solving this equation for s^* will give us the PSNE strategy profiles.

Example 1. Consider the simple case of a Cournot (quantity competition) setting where firms have identical linear costs (i.e. $c_i(q_i) = cq_i$) and demand is linear: $p = a - bQ$. Let's assume we have just 2 firms, and consider firm 1's best response function:

$$BR_1(q_2) = \arg\max_{q_1} [a - b(q_1 + q_2)]\, q_1 - cq_1.$$

The first-order condition tells us that $BR_1(q_2) = \frac{a-c}{2b} - \frac{q_2}{2}$. We can see that firm 2's problem is identical, so that firm's best response is $BR_2(q_1) = \frac{a-c}{2b} - \frac{q_1}{2}$.

A PSNE is a fixed point of the overall best-response function; that is, it is a solution to the following system of equations:

$$q_1 = \frac{a-c}{2b} - \frac{q_2}{2}$$
$$q_2 = \frac{a-c}{2b} - \frac{q_1}{2}$$

Solving this system of equations[1] yields the PSNE $q_1^* = q_2^* = \frac{a-c}{3b}$.

We can also use the best-response correspondences to help us find PSNE visually. If we have 2 players in a game, we can create a graph with player 1's strategies on one axis and player 2's strategies on the other. We then plot each player's best response(s) as a function of their opponent's strategies. Any point where the best responses overlap or intersect is a PSNE. This is because at these points, both players are mutually best responding.

We can see this approach in action with the Cournot example in Figure 1.

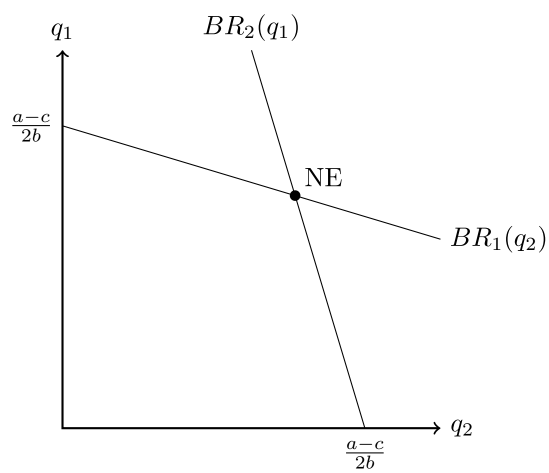

Figure 1: Seeing a PSNE graphically in the Cournot example. A NE is any point were the best response correspondences intersect.

2.2 Mixed Strategy Nash Equilibrium

2.2.1 Defining a Mixed Strategy Nash Equilibrium

As Doug noted in lecture, Pure Strategy Nash equilibria often do not exist. Often in these settings, randomization of strategies seems like a reasonable re-

[1]The symmetry of the problem allows us to take an algebraic shortcut: Since the firms' problems are identical, a symmetric PSNE must exist. That allows us to look simply for q^* such that $q^* = \frac{a-c}{2b} - \frac{q^*}{2}$.

sponse. We formalize this intuition with a new solution concept: mixed strategy Nash equilibrium (MSNE), which we define in the following way.

Consider the normal form game $\{\{S_i\}_{i=1}^I, g\}$. Let $K_i = |S_i| < \infty$ enumerate player i's pure strategies. Then we can let Δ_i be the K_i-dimensional simplex, so that $\delta_i \in \Delta_i$ is the probability vector over i's pure strategies; that is, $\delta_{ik} = P(i \text{ plays pure strategy } s_i^k)$. We call a particular δ_i a **mixed strategy** of player i.

We then define $\Delta = \times_{i \in I} \Delta_i$ as the mixed strategy profile set, and $\delta \in \Delta$ as a mixed strategy profile. Then the function $\pi : \Delta \to \mathbb{R}^I$ assigns payoffs for each $\delta \in \Delta$ as follows:

$$\pi_i(\delta) = E_S[g_i(s)|\delta] = \sum_{s \in S} g_i(s) P(s|\delta).$$

That is, each δ gives a probability distribution over all possible pure strategy profiles, each of which corresponds to a payoff for player i. That player's payoff for δ is then just the expectation of these payoffs given by the implied probability distribution.

Using these definition, we now define a new game $\{\{\Delta_i\}_{i=1}^I, \pi\}$ from the original game $\{\{S_i\}_{i=1}^I, g\}$.

Definition 2. A **mixed strategy Nash equilibrium** of the game $\{\{S_i\}_{i=1}^I, g\}$ is a pure strategy Nash equilibrium of the game $\{\{\Delta_i\}_{i=1}^I, \pi\}$.

In the above definition, we've defined MSNE as a PSNE of a different game. We can also define a MSNE more directly:

Definition 3. A **mixed strategy Nash equilibrium** of the game $\{\{S_i\}_{i=1}^I, g\}$ is a mixed strategy profile $\delta^* \in \Delta$ such that

$$E_S[g_i(s)|\delta_i^*, \delta_{-i}^*] \geq E_S[g_i(s)|\delta_i, \delta_{-i}^*] \quad \forall \delta_i \neq \delta_i^* \quad \forall i \in I.$$

2.2.2 Finding a Mixed Strategy Nash Equilibrium

Finding the mixed strategy Nash equilibria of a game is made fairly straightforward by the following observation:

Theorem 1. *In a MSNE, each player must be indifferent between all pure strategies to which her mixed strategy attaches positive probability.*

Proof. Suppose that the mixed strategy profile δ^* is a MSNE. Note that we can write player i's payoffs in in the following way:

$$\begin{aligned} E_S[g_i(s)|\delta_i^*, \delta_{-i}^*] &= E_S[g_i(s)|\delta_i^*, \delta_{-i}^*] \\ &= E_{S_i}[E_{S_{-i}}[g_i(s_i, s_{-i})|\delta_{-i}^*]|\delta_i^*] \\ &= \sum_{k=1}^{K_i} \delta_{ik}^* E_{S_{-i}}[g_i(s_i^k, s_{-i})|\delta_{-i}^*]. \end{aligned}$$

Suppose without loss of generality that i is playing s_i^1 and s_i^2 in this MSNE, and that $E_{S_{-i}}[g_i(s_i^1, s_{-i})|\delta_{-i}^*] > E_{S_{-i}}[g_i(s_i^2, s_{-i})|\delta_{-i}^*]$. That is, given the other players are playing according to δ^*, player i is *not* indifferent between s_i^1 and s_i^2. But then we have the following:

$$\begin{aligned} &E_S[g_i(s)|\delta_i^*, \delta_{-i}^*] \\ &= \delta_{i1}^* E_{S_{-i}}[g_i(s_i^1, s_{-i})|\delta_{-i}^*] + \delta_{i2}^* E_{S_{-i}}[g_i(s_i^2, s_{-i})|\delta_{-i}^*] + \ldots + \delta_{iK_i}^* E_{S_{-i}}[g_i(s_i^{K_i}, s_{-i})|\delta_{-i}^*] \\ &< \delta_{i1}^* E_{S_{-i}}[g_i(s_i^1, s_{-i})|\delta_{-i}^*] + \delta_{i2}^* E_{S_{-i}}[g_i(s_i^1, s_{-i})|\delta_{-i}^*] + \ldots + \delta_{iK_i}^* E_{S_{-i}}[g_i(s_i^{K_i}, s_{-i})|\delta_{-i}^*] \\ &= (\delta_{i1}^* + \delta_{i2}^*) E_{S_{-i}}[g_i(s_i^1, s_{-i})|\delta_{-i}^*] + \ldots + \delta_{iK_i}^* E_{S_{-i}}[g_i(s_i^{K_i}, s_{-i})|\delta_{-i}^*] \\ &= E_S[g_i(s)|\delta_i, \delta_{-i}^*] \end{aligned}$$

for some $\delta_i \neq \delta_i^*$. Thus δ_i^* is not a best response, and so δ^* is not a MSNE. □

Note an important implication of this result: It is *your* indifference condition that ties down *your opponent's* probabilities in a MSNE. We see this at work in the following simple example.

Example 2. Consider the game Bach or Stravinsky, whose normal form is given in Figure 2. We can see that the game has two PSNE: (B, B) and (S, S).

	B	S
B	$(2, 1)$	$(0, 0)$
S	$(0, 0)$	$(1, 2)$

Figure 2: The normal form of the game Bach or Stravinsky.

Next, let us consider possible MSNE. We can parameterize each player's mixed strategies as follows: Let $p = P(s_1 = B)$ and $q = P(s_2 = B)$. Then a mixed strategy profile is of the form $((p, 1-p), (q, 1-q))$.

It is important to note that there can be no MSNE where 1 player mixes but the other player plays a pure strategy. This is because given a player's pure strategy, his opponent has a unique best response. Therefore, we look for MSNE where both players are strictly mixing.

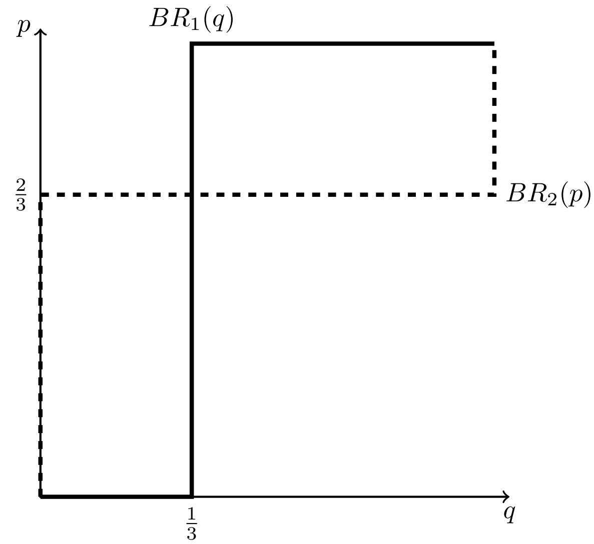

Figure 3: Seeing a the MSNE of Bach or Stravinsky.

Given player 2's mixing, player 1 must be indifferent between playing B and playing S:

$$\begin{aligned} E[g_1(B)|q] &= E[g_1(S)|q] \\ 2q + 0(1-q) &= 0q + 1(1-q) \\ q &= \frac{1}{3} \end{aligned}$$

And given player 1's mixing, player 2 must be indifferent between playing B and playing S:

$$\begin{aligned} E[g_2(B)|q] &= E[g_2(S)|p] \\ p + 0(1-p) &= 0p + 2(1-q) \\ p &= \frac{2}{3} \end{aligned}$$

Thus $\big((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\big)$ is a MSNE.

For the visually inclined, it can be very helpful to draw the best-responses for both players in $p-q$ space. It is straightforward to check that 1's best-response to q is as follows:

$$p = BR_1(q) = \begin{cases} 0 & \text{if } 0 \leq q < \frac{1}{3} \\ \in [0,1] & \text{if } q = \frac{1}{3} \\ 1 & \text{if } \frac{1}{3} < q \leq 1. \end{cases}$$

Note that as we showed above, when $q = \frac{1}{3}$, 1 is indifferent between his two pure strategies, and thus between any mixture of the two as well. Similarly, we can find 2's best response:

$$q = BR_1(p) = \begin{cases} 0 & \text{if } 0 \leq p < \frac{2}{3} \\ \in [0,1] & \text{if } p = \frac{2}{3} \\ 1 & \text{if } \frac{2}{3} < p \leq 1. \end{cases}$$

If we graph these two best response curves, we can see that all the MSNE are given by the intersections of the two curves. Of course, a PSNE is a special case of MSNE.

3 Problems

Problem 1. *War of attrition (Osbourne & Rubinstein 18.5)*

Two players are involved in the dispute over an object. The value of the object to player i is $v_i > 0$. Time is a continuous variable that starts and 0 and runs indefinitely. Each player chooses when to concede the object to the other player. If the first player to concede does so at time t, the other player receives the object at that time. If both players concede at the same time, the object is split equally between them, player i receiving a payoff of $\frac{v_i}{2}$. Time is valuable: until the first concession both players lose one unit of payoff per unit time.

(a) Formulate this situation to a a strategic game. What are the strategy sets of the players? What are the payoff functions?

(b) Write down and draw the best-response correspondences for each player.

(c) Show that in all PSNE of this game, one player concedes immediately.

Problem 2. *A game in extensive form.*

Consider the game shown in Figure 4.

(a) What is the normal form of this game?

(b) Find all MSNE of this game.

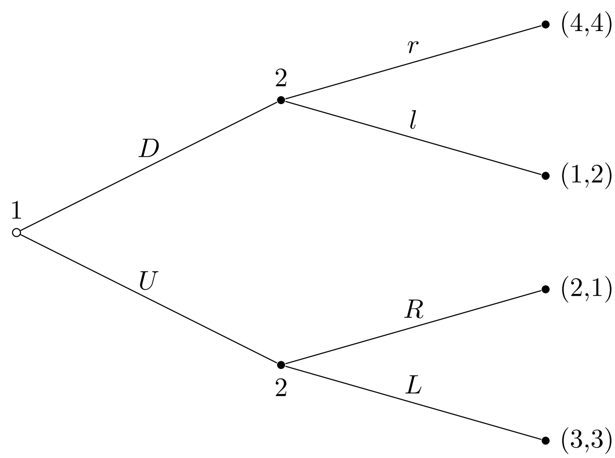

Figure 4: The game for problem 1.

4 Solutions

Solution 1.

(a) The players's stategies are to name $t_i \in [0, \infty)$. The payoff function is as follows:

$$g_i(t_i, t_j) = \begin{cases} -t_i & \text{if } t_i < t_j \\ \frac{v_i}{2} - t_i & \text{if } t_i = t_j \\ v_i - t_j & \text{if } t_i > t_j \end{cases}$$

(b) If my opponent is waiting longer than my valuation of the object, I don't want to hold out to win, since this will guarantee a negative payoff. I even don't want to try to tie my opponent, since this also guarantees a negative payoff. My best response is in fact to concede immediately. If my opponent is waiting exactly my valuation, I can get a payoff of zero by conceding immediately, or by waiting longer than my valuation. Any other waiting time will give a negative payoff. Lastly, if my opponent gives up before my valuation, I can guarantee myself positive payoffs by waiting just longer than him. Thus my best response function is as follows:

$$BR_i(t_j) = \begin{cases} 0 & \text{if } t_j > v_i \\ \{0\} \cup \{t | t \in (t_j, \infty)\} & \text{if } t_j = v_i \\ \{t | t \in (t_j, \infty)\} & \text{if } t_j < v_i \end{cases}$$

(c) By drawing the best response functions of each player on the same graph, it should be clear that we have two types of PSNE: $(t_1 = 0, t_2 \geq v_1)$ and $(t_1 \geq v_2, t_2 = 0)$. In both of these types of PSNE, one player concedes immediately.

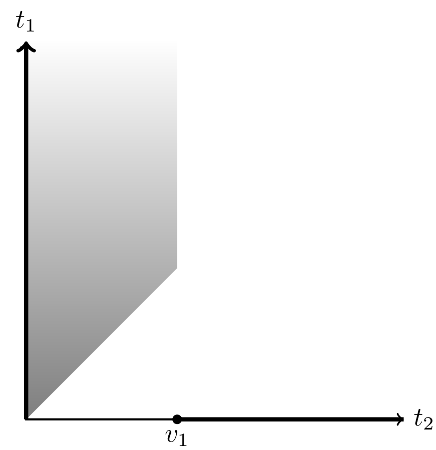

Figure 5: The shaded area and thick lines indicate the best response correspondence for player 1.

Solution 2.

(a) The normal form is given in Figure 6.

	Ll	Lr	Rl	Rr
U	$(3,3)$	$(3,3)$	$(2,1)$	$(2,1)$
D	$(1,2)$	$(4,4)$	$(1,2)$	$(4,4)$

Figure 6: The normal form of the game in problem 1.

(b) We start by looking for any strategies that are removed by IDDS, as these will not be played with positive probability in a MSNE.[2] We can see that Rl is dominated by Lr for player 1, and unfortunately this is as far as we can get with IDDS. You can confirm this by showing that all other strategies are rationalizable. (Remember the set of rationalizable strategies is the same as the set of strategies surviving IDDS for two-player games.)

Next, you can easily check that there are 3 PSNE: (U, Ll), (D, Lr), and (D, Rr).

Lastly, it remains to find the MSNE. Note that if 1 is putting positive probability on U and D, then 2's best response is to play Lr only. To see this, note that if 1 is putting probability p on U, then 2's payoff from Lr is strictly higher than her payoff from Ll or Rr: $3p+4(1-p) > 3p+2(1-p)$ and $3p + 4(1-p) > 1p + 4(1-p)$. But if 2 is playing only Lr, then

[2]This is not mentioned in the lecture notes, but you can use it without proof in problem sets and on the exam.

1's unique best-response is to play D only. This is a contradiction, so no MSNE can involve 1 strictly mixing. Thus we just need to check two case.

Case 1: 1 plays U only. In this case, 2 is indifferent between playing Ll and playing Lr, so any mixture of the two is a best response for her. By that mixture must be such that 1 playing U is a best response as well. So, we need the payoff from U to be at least as big as the payoff from D, given that 2 is putting probability q on playing Ll. That is,

$$3q + 3(1 - q) \geq 1q + 4(1 - q),$$

which reduces to $q \geq \frac{1}{3}$. Thus $((1, 0)(q, 1 - q, 0, 0))$ is a MSNE for any $q \geq \frac{1}{3}$.

Case 2: 1 plays D only. In this case, 2 is indifferent between playing Lr and playing Rr, so any mixture of the two is a best response for her. By the same logic as above, we need the payoff from D to be at least as big as the payoff from U, given that 2 is putting probability q on playing Lr. That is,

$$4q + 4(1 - q) \geq 3q + 2(1 - q),$$

which is satisfied for any valid q. Thus $((0, 1)(0, q, 0, 1 - q))$ is a MSNE for any $q \in [0, 1]$.

Note that the PSNE we found above are special cases of the two families of MSNE.