# Economics 203: Section 1

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## 1 Logistics

### 1.1 Contact Information

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Please help out my email load by putting "203" somewhere in your subject line when sending me a message.

### 1.2 Section Meetings

Time: Tuesdays, 9:00 am to 10:50 am
Location: Education 313

Sections will begin with a quick discussion of key topics from the previous week. I will usually have a few comments prepared, but this is also a great time to ask me questions. If possible, try to email me your questions by the day before section, so that I have time to prepare a thorough answer. We will then dedicate the rest of the section to solving problems.

### 1.3 Resources

- Doug's lecture notes, posted on Coursework[1]
- My section notes, posted on Coursework and my personal page[2]
- The problem bank, posted on Coursework

[1] See `https://coursework.stanford.edu/portal/site/W14-ECON-203-01`
[2] See `http://stanford.edu/~jnaecker/teaching/203_W14.html`

- Problem bank solutions, posted on Coursework after problem sets turned in
- In addition to the textbooks in Doug's syllabus, I recommend Osbourne and Rubinstein's *A Course in Game Theory* for a more theoretical treatment of the material.

### 1.4 Problem Sets

- Problem sets will be assigned by me on Fridays, to be turned in the following Friday at 5:00 pm in my box on the second floor of the Landau building.
- Problem sets will consist of a subset of the problem bank problems. I will try to choose the problems so that the requisite material will be covered in the previous week's lectures, though sometimes the material will spill into the lecture on the Tuesday of the week the problems are due. I will try to choose problems so that a typical student should not need to spend more than 10 hours on a problem set.
- Problem sets are graded on a binary basis. You will get a 1 if you turn in the problem set on time, and you have written down an answer for each part of each question (correct or not). Otherwise you will get a 0 for that week. It is totally OK if the answer is wrong, as long as it looks like you attempted the given problem. It is also OK if you copy word-for-word from a previous year's solutions.

## 2 Review

### 2.1 Course Overview

The sections of the course are organized around two key concepts:

- **Static games** are ones in which we think of the players moving simultaneously; we model these with the normal form. **Dynamic games** are ones in which we think of the players moving sequentially; we model these with the extensive form.
- In games of **complete information**, all players know the payoffs of all other players. If this is not true, then we have a game of **incomplete information**.

This reveals the logic of the seven sections of Doug's notes:

1. Strategic environments: describe how we model social interactions

2. Static + complete information
3. Static + incomplete information
4. Dynamic games overview
5. Dynamic + complete information
6. Repeated games (a special case of dynamic + complete info)
7. Dynamic + incomplete information (e.g., signaling games)

## 2.2 Why Solution Concepts?

Once we have written down a game in either extensive or normal form, we can see all of the possible outcomes of the games. There may be a large or even infinite number of outcomes, so our job as game theorists is to predict which outcomes are likely to happen. Hopefully we can get a unique prediction, but at the very least we hope to identify a very small subset of all the outcomes. This is what we mean by a **solution**. Thus, a **solution concept** is a rule that, given the game, predicts which outcome or outcomes will actually come to pass.

## 2.3 Solution Concept: Dominant Strategy Solution

First, let's remind ourselves of the key notation for games in normal form:

- $\mathcal{I} = \{1, 2, \ldots, I\}$ is the set of players.
- A **strategy** for player $i$ is a mapping from that player's information sets to their feasible actions. If the number of information sets is small, then the strategy $s_i$ is conventionally written as a vector, or even just as a string of letters (eg $HH$ for matching pennies).
- A **strategy profile** $s$ is a vector listing all the strategies of all the players. The set of all possible strategy profiles is $S = S_1 \times S_2 \times \cdots \times S_I$. We usually write $S_{-i}$ to indicate the strategy set of all players but $i$.
- The function $g : S \to \mathbb{R}^I$ is called the **payoff function**.

**Definition 1.** A strategy $s_i$ is a **dominant strategy** for player $i$ iff for all $\hat{s}_i \in S_i$ such that $\hat{s}_i \neq s_i$ and for all $s_{-i} \in S_{-i}$, we have $g_i(s_i, s_{-i}) > g_i(\hat{s}_i, s_{-i})$.

In plain English: A dominant strategy is one that is the best response for a player for every possible strategy their opponent is playing.

**Definition 2.** If all players have a dominant strategy, we say that the game has **dominant strategy solution**.

**Example 1.** The game Prisoner's Dilemma (Figure 1) has a dominant strategy solution. To see why, note that if player 2 is playing *Fink*, then palyer 1's best

| | $NF$ | $F$ |
|---|---|---|
| $NF$ | $(-2,-2)$ | $(-5,-1)$ |
| $F$ | $(-1,-5)$ | $(-4,-4)$ |

Figure 1: Prisoner's Dilemma. Player 1 is the row player and player 2 is the column player.

response is *Fink* (since $-4 > -5$). If player 2 is playing *Not Fink*, player 1's best response is still *Fink* (since $-1 > -5$). It is easy to see this graphically by plotting the payoff to each of player 1's strategies as a function of player 2's strategy, as in Figure 2. Thus *Fink* is a dominant strategy for player 1. Similarly, player 2's dominant strategy is also *Fink*. Thus the dominant strategy solution is (*Fink*, *Fink*).

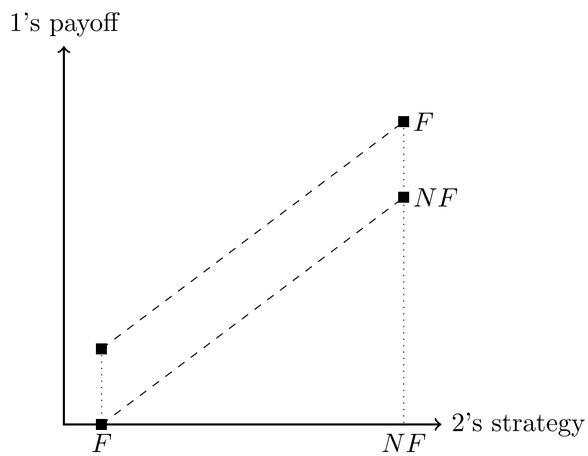


Figure 2: Seeing a dominant strategy graphically: note that the payoff for player 1 of playing *Fink* is always strictly above that of playing *Not Fink*.

## 2.4 Solution Concept: Dominance Solvable

For this solution concept, we need to defined a mixed strategy, even though we haven't formally talked about them in class yet.

Let $K_i = |S_i|$ for each player $i$ be the size of that player's strategy set. Label the $K_i$ strategies for $i$ by $k = 1, 2, \ldots, K_i$. A probability measure $\rho$ on the strategy set $S_i$ is a vector $\rho = (\rho_1, \ldots, \rho_{K_i})$ such that $\rho_k \geq 0$ for all $1 \leq k \leq K_i$ and $\sum_{l=1}^{K_i} \rho_k = 1$.

**Definition 3.** A strategy $s_i^k$ is a **dominated strategy** for player $i$ iff there exists a measure $\rho$ on $S_i$ such that

$$\sum_{m=1}^{K_i} \rho_m g_i(s_i^m, s_{-i}) > g_i(s_i^k, s_{-i})$$

for all $s_{-i} \in S_{-i}$.

In plain English: A strategy is dominated if the player can do strictly better with some (possibly degenerate) mixture of strategies, for every possible strategy their opponent could be playing.

**Definition 4.** If iterative deletion of dominated strategies yields a unique outcome, we say that the game is **dominance solvable**.

*Note.* In IDDS, we delete only one player's strategy at a time, even if at a given round multiple players have dominated strategies.

**Example 2.** Consider the normal form game from Doug's notes, reproduced in Figure 3. It is clear that both players do not have a dominant strategy. Note, however, that 2's strategy $c$ is strictly dominated by playing and even mixture of $a$ and $b$. This is easily seen graphically in Figure 4. Deleting this strategy begins a chain of IDDS that eventually leads to the outcome $(A, a)$.

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $A$ | $(4, 10)$ | $(3, 0)$ | $(1, 3)$ |
| $B$ | $(0, 0)$ | $(2, 10)$ | $(10, 3)$ |

Figure 3: A normal form game that is dominance solvable.

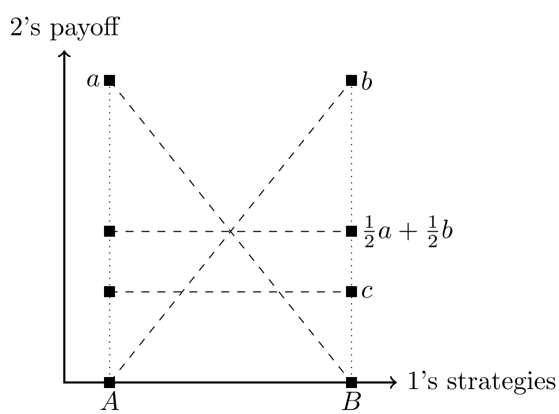


Figure 4: Note that the mixture of strategies $a$ and $b$ dominates strategy c for player 2.

This example highlights the fact that the payoffs listed are the von Neumann-Morgenstern (vNM) utilities for the given outcome. This means that the risk aversion for each player, if any, is already built into the payoff numbers, and so players rank outcomes or mixtures of outcome based solely on their expected value.

### 2.4.1 A checklist

How do you quickly tell whether a strategy $s_i$ is or is not dominated? Here is a checklist:

1. Is there a pure strategy $\hat{s_i} \neq s_i$ that dominates $s_i$? If so, then $s_i$ is dominated.
2. Is $s_i$ the (not necessarily unique) best response to some $s_{-i}$? If so, then $s_i$ is *not* dominated.
3. Is there a mixture of other strategies for $i$ that dominates $s_i$? If so, then $s_i$ is dominated.

The last item is just the definition of $s_i$ being a dominated strategy. My point here is simply that you don't have to immediately start looking for some complicated mixture of strategies to tell whether or not a strategy is dominated. The first two items on the checklist may give you a very quick answer.

### 2.4.2 Dealing with mixed strategies

But what happens if you get all the way to item 3 on the checklist, and a dominating mixture doesn't jump out at you? One strategy is to start writing down constraints on $\rho$, the vector of probability weights, that the definition of dominance requires.

**Example 3.** Consider the normal form game in Figure 3 from above. The checklist quickly establishes that for player 2, $a$ and $b$ are not dominated (since they are best responses to player 1 choosing $A$ and $B$, respectively). But items 1 and 2 of the checklist fail to resolve the status of $c$. So, how did we know that $c$ was dominated by a mixture of $a$ and $b$?

Note that the definition of a strictly dominated strategy tells us that we are looking for $\rho_a$ and $\rho_b$ such that

$$\begin{aligned} &\rho_a \cdot 10 + \rho_b \cdot 0 > 3 \qquad \text{and} \\ &\rho_a \cdot 0 + \rho_b \cdot 10 > 3, \end{aligned}$$

which imply that $\rho_a > \frac{3}{10}$ and $\rho_b > \frac{3}{10}$. From there, it is easy to pick a combination of $\rho_a$ and $\rho_b$ that dominates $c$.

Suppose instead that we were facing the game given in Figure 5. Again $a$ and $b$ are not dominated. This time, however, we find that a dominating mixture

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $A$ | $(4, 10)$ | $(3, 0)$ | $(1, 6)$ |
| $B$ | $(0, 0)$ | $(2, 10)$ | $(10, 6)$ |

Figure 5: A normal form game with payoffs in the last column modified.

requires $\rho_a > \frac{6}{10}$ and $\rho_b > \frac{6}{10}$. But this contradicts $\rho_a + \rho_b = 1$. Thus in this game $c$ is not dominated.

# 3 Problems

## 3.1 Split-The-Pie

Consider the following situation. Two people ($A$ and $B$), want to split a delicious pie. $A$ cuts the pie into two pieces, and then $B$ picks one of those two pieces to consume. $A$ gets to consume the other piece. For this problem, let's assume that $A$ can only cut the pie in half, or not cut it at all. Let's also assume that each player's vNM utility is equal to the fraction of the pie they consume.

1. Draw the extensive form for this game. (You can assume the pie is of size 1.)
2. What are the strategy sets for the two players?
3. Draw the normal form of this game.
4. Are there any dominated strategies? Is the game dominance solvable?

## 3.2 Split-The-Pie Redux

Consider the situation in 3.1 above, but now $A$ can cut the pie into two pieces any way she wants.

1. Draw the extensive form for this game.
2. What are the strategy sets for the two players?
3. Are there any dominated strategies? Is the game dominance solvable?

## 3.3 Guessing Game

There is a class of simultaneous-move games known as *two-person guessing games*. In these games, each player's move is a guess $x_i \in [a_i, b_i]$. Each player has a *target*, $t_i = \alpha_i x_j$, where $\alpha_i$ is called $i$'s *multiplier*. Payoff for player $i$ is

$-|x_i - t_i|$. That is, each player wants guess as close as possible to their multiplier times the other player's guess.

For this problem, assume $a_1 = a_2 = 0$, $b_1 = b_2 = 100$, and $\alpha_1 = \alpha_2 = \frac{1}{4}$.

1. Do the players have any dominated strategies?
2. Is the game dominance-solvable?

# 4 Solutions

## 4.1 Split-The-Pie

1. We have two players, $A$ and $B$. Let's say that the pie is cut into two pieces, $Y$ and $Z$. Then $A$'s choice is what size to make piece $Y$, after which $B$'s choice is whether to pick piece $Y$ or piece $Z$. Thus the extensive form of the game can be drawn as in Figure 6. Note that the extensive form tells us the game tree, any information sets, and the payoffs. The set of players is implicit in both the tree and the vector listing the payoffs.

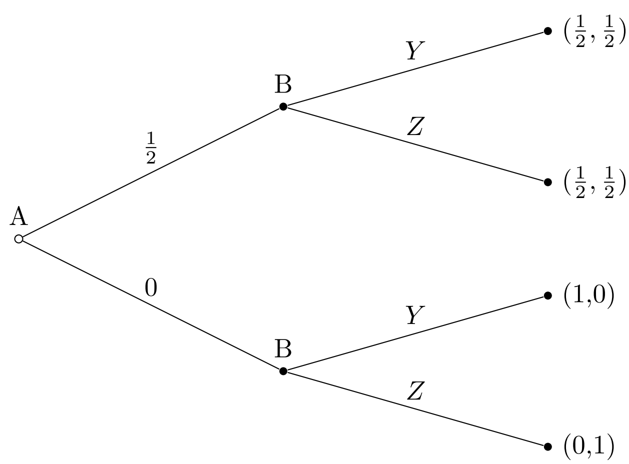


Figure 6: The extensive form of Split-The-Pie.

2. Player $A$ has one information set, so in this case actions and strategies will coincide. Thus $S_A = \{0, \frac{1}{2}\}$.

   Player $B$ has two information sets: one after $A$ plays $\frac{1}{2}$ and one after $A$ plays 0. Thus each member of $S_B$ will be an ordered list: the first term will be $B$'s action after the first information set, and the second term will be the action to follow the second information set. Thus $S_B = \{(Y,Y),(Y,Z),(Z,Y),(Z,Z)\} = \{Y,Z\} \times \{Y,Z\}$.

3. The normal form is given in Figure 7. Player $A$ is the row player and player $B$ is the column player. Note that the normal form tells us all possible strategies for all players, as well as the payoffs. As in the extensive form, the set of players is implicit in the payoffs (and in the fact that we can write the game as a matrix).

| | $(Y,Y)$ | $(Y,Z)$ | $(Z,Y)$ | $(Z,Z)$ |
|---|---|---|---|---|
| $\frac{1}{2}$ | $(\frac{1}{2},\frac{1}{2})$ | $(\frac{1}{2},\frac{1}{2})$ | $(\frac{1}{2},\frac{1}{2})$ | $(\frac{1}{2},\frac{1}{2})$ |
| $0$ | $(1,0)$ | $(0,1)$ | $(1,0)$ | $(0,1)$ |

Figure 7: The normal form of Split-The-Pie.

4. $A$ does not have a dominated strategy. To see why, note that 0 is not dominated because it is the best response when $B$ plays $(Y,Y)$, for example. Similarly, $\frac{1}{2}$ is not dominated because it is the best response when $B$ plays $(Z,Z)$, for example

   $B$ does not have a dominated strategy either, since all strategies give the same payoff when $A$ plays $\frac{1}{2}$.

   Thus, there are no dominated strategies and the game is not dominance solvable.

## 4.2 Split-The-Pie Redux

1. Let's say that $A$ chooses $x$, the size of piece $Y$. Then the extensive form can be drawn as in Figure 8.

2. $A$ can choose to cut the pie in any size, so $S_A = [0,1]$. $B$'s strategy is a little more complicated: for any split that $A$ gives him, he must say whether he would like piece $Y$ or piece $Z$. Thus, a strategy for $B$ is a function from the inteval $[0,1]$ to the set $\{Y,Z\}$. That is, $S_B$ is the set of all functions $f$ such that $f : [0,1] \to \{Y,Z\}$.

3. There are no dominated strategies for $A$. To see why, suppose the some strategy $\bar{x} \in [0,1]$ is dominated. Then from the definition of a dominated strategy, there exists a measure $\rho$ on $[0,1]$ such that

$$\int \rho(x) g_A(x,Y) dx > g_A(\bar{x},Y) \tag{1}$$

   and

$$\int \rho(x) g_A(x,Z) dx > g_A(\bar{x},Z). \tag{2}$$

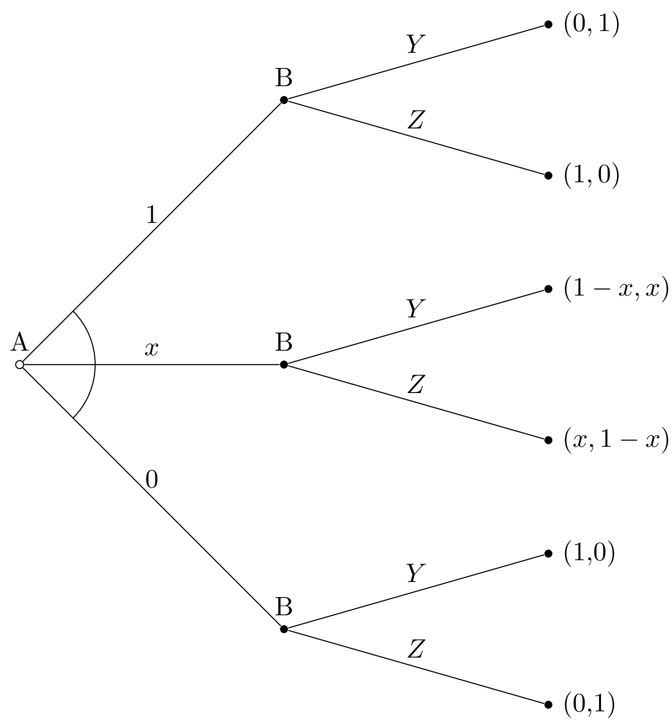


Figure 8: The extensive form of Split-The-Pie Redux.

But since $g_A(x, Y) = (1 - x)$ and $g_A(x, Z) = x$, we have the following implications. From (1),

$$\begin{aligned}\int \rho(x) g_A(x, Y) dx > g_A(\bar{x}, Y) &\Rightarrow \int \rho(x)(1-x)dx > (1-\bar{x}) \\ &\Rightarrow 1 - \int \rho(x) x dx > (1-\bar{x}) \\ &\Rightarrow \int \rho(x) x dx < \bar{x}\end{aligned}$$

But from (2),

$$\int \rho(x) g_A(x, Z) dx > g_A(\bar{x}, z) \Rightarrow \int \rho(x)(x) dx > \bar{x},$$

which is a contradiction. This $\bar{x}$ must not be dominated.

There are also no dominated strategies for $B$. If $A$ plays $x = \frac{1}{2}$, then $B$ is indifferent between all strategies.

Thus, there are no dominated strategies, and the game is not dominance solvable.

### 4.3 Guessing Game

1. Playing $x_i > 25$ is dominated by playing 25 for $i \in \{1, 2\}$. Proof:

   Let $x_i > 25$. Claim: $x_i$ dominated by $\bar{x_i} = 25$. From the definition of a dominated strategy it then must be that,

   $$\int \rho(x_j)\left(-|25 - \alpha_i x_j|\right) dx_j > \int \rho(x_j)\left(-|x_i - \alpha_i x_j|\right) dx_j.$$

   Note that both terms inside the absolute values must be positive, so we can drop the absolute values:

   $$\begin{aligned} -\int \rho(x_j)\left(25 - \alpha_i x_j\right) dx_j &> -\int \rho(x_j)\left(x_i - \alpha_i x_j\right) dx_j \\ \int \rho(x_j)\left(25 - \alpha_i x_j\right) dx_j &< \int \rho(x_j)\left(x_i - \alpha_i x_j\right) dx_j \\ 25 - \int \rho(x_j)\left(\alpha_i x_j\right) dx_j &< x_i - \int \rho(x_j)\left(\alpha_i x_j\right) dx_j \\ 25 &< x_i \end{aligned}$$

   This last line is true by assumption, so $x_i > 25$ is in fact dominated!

2. In the first round, 1 eliminates strategies above $25 = \left(\frac{1}{4}\right) 100$. In the second round, 2 can then eliminate strategies greater than $\left(\frac{1}{4}\right)^2 100$. In the third round, 1 eliminates strategies greater than $\left(\frac{1}{4}\right)^3 100$, and so on. After an infinite number of rounds, all positive strategies are eliminated for both players. Thus the dominance solvability outcome is $x_1 = x_2 = 0$.