Economics 203: Section 8

Jeffrey Naecker

March 6, 2012

1 Logistics

A few brief notes:

- Apologies for getting behind on the grading again, but Problem Sets 6 will be returned by next Tuesday's section.
- Your final in on the evening of Wednesday, March 21st. We will need to schedule a review session, so start thinking about what day and time you would like to have it.

2 Concepts

We have no new solution concepts this week, so I will just briefly summarize the applications that were discussed in lecture.

2.1 Sequential Bargaining

We imagine that two player are splitting a pie of size 1. They take turns making offers, which the other player can then accept or reject. If the split $(x, 1-x)$ is implemented in period t, then the payoffs are $(\delta_1^t x, \delta_2^t (1-x))$.

Note that in both the finite and infinite versions of this game, the SPNE outcome is such that the split is agreed upon without delay. This is not, however, what happens in the lab. In fact, we often see a strong deadline effect, where players do not reach agreement until the last few seconds of the game. We even see that players sometimes do not reach an agreement at all.

2.2 Repeated Games

We consider the case where a so-called **stage game** is repeated some number of times. In this application, we have three important variables:

- The horizon: Finite or infinite
- The payoffs (if infinite): discounted, average payoff criterion, or overtaking criterion
- The solution concept: Nash equilibrium or subgame perfect Nash equilibrium

For infinitely repeated games, there are a few key points to remember:

- Playing a Nash equilibrium of the stage game in every period is an equilibrium of the repeated game.

- When using discounted payoffs: Players can usually support playing cooperatively in SPNE by using a **Nash reversion** strategy, wherein if any player deviates from the cooperative outcome, play reverts to a NE of the stage game. Such a SPNE will usually only hold for certain δ. Note that for such constructions, you only need to check against deviations from the equilibrium path. Off-path, we are playing a stage-game NE each period, which by the above point is a NE of the subgame, and thus can be part of a SPNE for the repeated game as a whole.

- Consider a strategy profile δ of the stage game. For each player i, we can define that player's **min-max payoff** $\pi_i^m = \min_{\delta_{-i}} \max_{\delta_i} \pi_i(\delta)$.[1] Consider a payoff profile that is in the convex hull of stage game NE payoffs and also has no player receiving worse than his min-max payoff. Then the **folk theorem** says that this payoff profile can be supported as the average payoffs for some NE of the repeated game.

3 Problems

Problem 1. *Problem bank #61.*

Suppose that there are I consumers, each with demand given by $q_i(p) = a - bp$ $(a, b > 0)$. Firms incur a cost K when they enter the industry, and a variable cost c for each unit produced. There are infinitely many (identical) potential firms.

(a) Suppose that all firms make entry and production choices (quantities) simultaneously in a single stage. What happens to equilibrium prices, output levels, and consumer surplus as I goes to infinity?

(b) How do your answers in part A change if firms first make entry decisions simultaneously, and then, having observed each others' entry decisions, choose quantities (assuming that the entry cost, K, is sunk)?

Problem 2. *Problem bank #65.*

Consider the bargaining model discussed in class, but instead of assuming that players discount future payoffs, assume that it costs $c < 1$ to make each offer. (Only the player making the offer incurs this cost, and players who have made offers incur this cost even if no agreement is ultimately reached.) Assume first the horizon is finite, of length T.

(a) What is the (unique) SPNE of this alternative model?

(b) What happens as T approaches infinity?

(c) Now suppose that the horizon is infinite. What are the subgame perfect Nash equilibria of this game?

Problem 3. *Problem bank #75.*

Consider the following game in normal form (where player 1 chooses rows, player 2 chooses columns, and player 1's payoff is given first in each cell).

	a	b	c
A	5,3	5,5	3,4
B	4,10	9,9	4,11
C	3,3	11,4	5,5

(a) Suppose that this game is repeated a finite number of times, and that the discount factor (δ) is unity. Characterize the set of subgame perfect equilibria.

[1] Intuitively, this is player $i's$ best response payoff when his opponents are out to get him, in the sense that they pick strategies to force him to his worst possible best-response.

(b) Suppose that the game is repeated infinitely, and that δ is less than unity. Suppose that the players try to sustain the cooperative choice (B, b) in every period through Nash reversion. For what values of δ is this an equilibrium? Is it subgame perfect?

Problem 4. *Problem bank #78.*

Consider the linear Cournot oligopoly model where price p is given as a linear function of total output Q, $p = a - bQ$, and where output is produced by each firm i at constant marginal cost c (same for all firms). Suppose there are N firms. Suppose also that the stage game will be repeated infinitely, and that firms discount future payoffs at the rate δ. Let δ_N be the minimum discount factor for which it is possible to sustain complete collusion in subgame perfect Nash equilibrium, using Nash reversion. Solve for δ_N as a function of N. How does it vary with N? Is this what you would expect? Why, or why not?

Problem 5. *Problem bank #84.*

Consider an infinitely repeated game with 2 firms. In each period, two firms simultaneously choose a characteristic of their products from a binary set $\{a, b\}$, if they choose the same characteristic, they split the market equally and each gets stage payoff of $\frac{\pi}{2}$, otherwise firm 1 always gets the entire market (so firm 1 gets stage payoff of π and firm 2 gets 0).

(a) What is the minmax payoff for each firm in the stage game?

(b) What is the highest payoff for each firm that can be an outcome of a Nash equilibrium of the infinitely repeated game?

4 Solutions

Solution 1.

(a) We can see that we will have three conditions for NE: that the firms entering to not want to exit, that the firms entering are choosing the optimal quantity, and than the non-entering firms to not wish to enter. We consider these three conditions in turn.

- Optimal quantity: Note that total demand in the economy is $Q = I(a - bp)$, or equivalently $p = \frac{a}{b} - \frac{Q}{bI}$. Thus a single firm's optimal quantity is given by

$$q_i = \arg\max_q \left(\frac{a}{b} - \frac{q + q_{-i}}{bI} - c \right) q,$$

  for which the FOC gives $\frac{a}{b} - c - 2\frac{q_i}{bI} - \frac{q_{-i}}{bI} = 0$. Noting that $q_{-i} = (N-1)q_i$ by symmetry, we find that $q_N^C = (a - bc)\frac{I}{N+1}$, where we include the N subscript to emphasize that this is the quantity when there are N firms in the market.

- Optimal entry: From the quantity above, we can calculate the market price: $p_N^C = \frac{a}{b} - \frac{1}{bI}\frac{N(a-bc)I}{N+1} = \frac{a}{b} - \frac{1}{b}\frac{N(a-bc)}{N+1}$. From here we can calculate the profit for the entering firms: $\pi_N^C = \left[\frac{a}{b} - \frac{1}{b}\frac{N(a-bc)}{N+1} - c\right](a-bc)\frac{I}{N+1} = \frac{(a-bc)^2 I}{(N+1)^2 b}$. For entering firms to not wish to deviate out of the market, we must have $\frac{(a-bc)^2 I}{(N+1)^2 b} - K \geq 0$, since they pay cost K for entering.

- Optimal non-entry: Note that if a non-entrant were to deviate to entering, that firm would face N other firms, all producing quantity q_N^C. Thus the deviating firm's optimal quantity would solve

$$q_D = \arg\max_q \left(\frac{a}{b} - \frac{q + N q_N^C}{bI} - c \right) q,$$

which we can solve to find that $q^D = \frac{(a-bc)I}{2}\frac{N}{N+1}$. Then the total quantity is $Q = Nq_N^C + q^D = N(a-bc)\frac{I}{N+1} + \frac{(a-bc)I}{2}\frac{N}{N+1} = \frac{N+2}{N}\frac{a-bc}{N+1}I$, and the new price is $p = \frac{a}{b} - \left(\frac{a}{b} - c\right)\frac{N+2}{2(N+1)}$. This finally allows us to calculate the profit for the deviating firm: $\pi^D = \frac{(ab-c)^2}{4b}\frac{1}{(N+1)^2}I$. The non-entering firm will thus prefer to stay out if $\frac{(ab-c)^2}{4(N+1)^2 b}I - K \leq 0$.

We are now asked to find the limits of some important values as $I \to \infty$.

- Note that for fixed K, the entry and non-entry conditions restrict the ratio of N and I to be bounded above and below. Thus as $I \to \infty$, $N \to \infty$ as well. Then $p_N^C = \frac{a}{b} - \frac{1}{b}\frac{N(a-bc)}{N+1} \to c$.
- Total output for is $Nq_N^C = (a-bc)\frac{IN}{N+1} \to \infty$.
- Total surplus is $CS = \frac{1}{2}\left(\frac{N}{N+1}\right)^2 \frac{(a-bc)^2}{b}I$. You can calculate this easily by noting that since demand is linear, surplus is just a triangle. Surplus goes to infinity, but note that surplus per consumer has a finite limit: $\frac{CS}{I} = \frac{1}{2}\left(\frac{N}{N+1}\right)^2 \frac{(a-bc)^2}{b} \to \frac{1}{2}\frac{(a-bc)^2}{b}$.

(b) Note that we still have three conditions for a SPNE: that entering firms pick optimal prices, that entering firms do not wish to exit, and that non-entering firms do not wish to enter. The first two conditions are the same as above, but the third condition changes, since now a firm deviating from non-entry to entry will be observed by its competitors before they all name prices. Thus this firm should pick the $N+1$ Cournot quantity, not the quantity we calculated above. Its profit upon entry will just be the $N+1$ Cournot profit. Thus the non-entry condition becomes $\frac{(ab-c)^2}{4(N+2)^2 b}I - K \leq 0$.

Note that the equilibrium path has not changed, so the limits calculate in the previous part are the same here.

Solution 2.

(a) Since the game is finite, we can solve by backwards induction. Throughout, let the x be the amount that player 1 gets in any split, so that the payoffs are $(x, 1-x)$ (not including any costs incurred). First, we assume that T is even.

Period T: Player 2 makes the offer in the final period. Player 1 will accept any offer, so 2 offers $x = 0$, which is accepted. Thus the payoffs *in this round only* are $(0, 1-c)$, since 2 incurs cost c for making an offer.[2]

Period T-1: Player 1 makes the offer, and 2 accepts or rejects. Note that 2's continuation payoff for rejecting is $1-c$, so he will accept iff $1-x \geq 1-c$. 1's continuation payoffs in this case will be 0, so his best response is to offer $1-x = 1-c$ to 2, which is accepted. Note that payoffs in this round are $(0, 1-c)$, since 1 gets c of the pie buy pays c for making the offer, netting him 0.

Period T-2: Player 2 makes the offer, and 1 accepts or rejects. Not that 1's continuation payoff for rejecting is 0, so he will accept any offer. Thus 2 proposes $x = 0$, which accepted. Yet again, payoffs in this round are $(0, 1-c)$.

We can now see that continuation payoffs will *always* be $(0, 1-c)$, so the steps above will repeat until the reach the beginning of the game. Thus we have that the following strategy profile is the unique SPNE of the game (when T even):

- Player 1 proposes $(c, 1-c)$ in every period where he gets to make an offer.
- Player 1 accepts an offer iff $x \geq 0$.

[2] All the costs up until this point are sunk, so we don't need to consider them when figuring out which actions the players will take in this subgame.

- Player 2 proposes $(0, 1)$ in every period where he gets to make an offer.
- Player 2 accepts an offer iff $1 - x \geq 1 - c$.

In this case, the game ends immediately, and payoffs are $(0, 1 - c)$.

Next, we consider the case of T odd. But note that by the same backwards induction argurment, we get that the roles of the players are just switched. Thus we have that the following strategy profile is the unique SPNE of the game (when T odd):

- Player 2 proposes $(1 - c, c)$ in every period where he gets to make an offer.
- Player 2 accepts an offer iff $1 - x \geq 0$.
- Player 1 proposes $(1, 0)$ in every period where he gets to make an offer.
- Player 1 accepts an offer iff $x \geq 1 - c$.

Again in this case, the game ends immediately, and payoffs are $(1 - c, 0)$.

(b) The outcome depends only on the parity of T, and so he payoffs do not converge as T goes to infinity. Compare this to the discounting case, where the payoffs do converge as T goes to infinity.

(c) Note that the solution to the finite horizon version was *stationary*; that is, the specified strategies did not depend on the period. Thus, we can guess that there is a stationary strategy for the infinite horizon version as well, which also ends the game immediately. Such an equilibrium would take the following form:

- Player 1 proposes $(a, 1 - a)$ in every period where he gets to make an offer.
- Player 1 accepts an offer iff $x \geq b$.
- Player 2 proposes $(b, 1 - b)$ in every period where he gets to make an offer.
- Player 2 accepts an offer iff $1 - x \geq 1 - a$.

For this proposed profile to be an equilibrium, we need the following: If player 1 rejects 2's offer of b, then he gets payoff $a - c$ in the next. Thus we must have $b \geq a - c$. Similarly, if 2 rejects 1's offer of $1 - a$, he gets $1 - b - c$ in continuation. Thus we must have $b \leq a - c$. Combining these results we find that $b = a - c$. Thus I propose that the following strategy profile is a SPNE, for any $a \in [c, 1]$:

- Player 1 proposes $(a, 1 - a)$ in every period where he gets to make an offer.
- Player 1 accepts an offer iff $x \geq a - c$.
- Player 2 proposes $(a - c, 1 - a + c)$ in every period where he gets to make an offer.
- Player 2 accepts an offer iff $1 - x \geq 1 - a$.

To check that this is in fact a SPNE, it suffices to check one-shot deviations by each player.

- In equilibrium, 1 offers $(a, 1 - a)$, which is accepted, giving him a payoff of $a - c$. If 1 were to increase his offer to 2, 2 would accepts and 1 would get a lower payoff. If 1 were to decrease his offer, 2 would reject, and in the ensuing period 1 would accept 2's offer of $a - c$. 1's total payoffs would be $a - c - c < a - c$, so this deviation is not profitable. Thus 1 will not deviate from his equilibrium offer.
- In equilibrium, 2 offers $(a-c, 1-a+c)$, which is accepted, giving him a payoff of $1-a+c-c = 1-a$. If 2 were to increase his offer to 1, 1 would accepts and 2 would get a lower payoff. If 2 were to decrease his offer, 1 would reject, and in the ensuing period 2 would accept 1's offer of $1 - a$. 2's total payoffs would be $1 - a - c < 1 - a$, so this deviation is not profitable. Thus 2 will not deviate from his equilibrium offer.

- Consider 1's accept/reject decision. If he rejects, he gets payoff $a - c$ in the ensuing round. If he accepts some offer x he gets payoff x. Clearly he should accept iff $x \geq a - c$, as specified in the equilibrium.
- Consider 12s accept/reject decision. If he rejects, he gets payoff $1 - a + c - c = 1 - a$ in the ensuing round. If he accepts some offer $1 - x$ he gets payoff $1 - x$. Clearly he should accept iff $1 - x \geq 1 - a$, as specified in the equilibrium.

No player has incentive to deviate, so the strategies above constitute a SPNE.

Solution 3.

(a) Note that the unique NE of the stage game is (C, c). Thus the only possible SPNE of the finitely repeated game is that 1 plays C in every stage and 2 place c in every stage.

(b) Nash reversion implies for following strategies. Players 1 and 2 start out by playing B and b, respectively. If the outcome is every previous period was (B, b), the players continue to play B and b. Otherwise, they play C and c in every period thereafter.

On the equilibrium path, the agents each get payoff 9 every period. Assuming 1 is playing B, 2 would like to deviate to c for a payoff of 11. Similarly, assuming 2 is playing b, 1 would like to deviate to playing C, also for a payoff of 11. To sustain cooperation, we thus require that

$$\frac{1}{1-\delta}9 \geq 11 + \frac{\delta}{1-\delta}5,$$

which is equivalent to $\delta \geq \frac{1}{3}$.

This is indeed subgame perfect. Note that the condition above ensures that no player wants to deviate from the equilibrium path. Off the path, the specific strategies are NE of the stage game, and thus no player has incentive to deviate there either.

Solution 4.

Nash reversion specifies the following strategies in the repeated game: Each firm starts by producing the collusion quantity in the first repetition of the game. If in all previous period, all firms have played that quanitity, all firms continue to play that quantity in the current period. Otherwise, all firms play the Cournot quantity for all remaining repetitions of the stage game.

First, some notation: In complete collusion, all firms produce identical amounts such that the total quantity is the monopoly quantity, Q^M. Let the profits for each firm in this case be π^M. If one firm deviates, they will best-respond to all other firms producing q^M. That deviating firm will get profit π^D. From then on, all firms produce q^C, the Cournot quantity, for profits π^C.

For a single-shot deviation not to be profitable, we require the following: The profits from playing the collusion quantity forever must be greater than the profits from getting the deviation profit one period, followed by the Cournot profit in every following period. That is, we need

$$\frac{1}{1-\delta}\pi^M \geq \pi^D + \frac{\delta}{1-\delta}\pi^C.$$

It remains to find π^M, π^D, and π^C, and then solve for δ.

- First, we find the collusion profits. Monopoly quantity is given by $Q^M = \arg\max_Q (a - bQ - c)Q$, which we can solve to find $Q^M = \frac{a-c}{2b}$. Then each colluding firm produces $q^M = \frac{a-c}{2bN}$. Plugging in, we find price is given by $p^M = a - b\frac{a-c}{2b} = \frac{a+c}{2}$. Thus, collision profits are $\pi^M = (p^M - c)q^M = \left(\frac{a+c}{2} - c\right)\frac{a-c}{2bN}$. This gives $\pi^M = \frac{(a-c)^2}{4bN}$.

- In the Cournot equilibrium, all firms must be best-responding to the quantity produced by their competitors. Let q_i be firm i's quantity, and q_{-i} be the total quantity produced by that firm's competitors. Then $q_i^C = \arg\max_q \left[a - b(q + q_{-i}) - c\right] q$. The FOC give $a - 2bq - bq_{-i} - c = 0$. But we know that by all firms are producing the same quantity, so $q_{-i} = (N-1)q$. Thus we find $q^C = \frac{a-c}{b(N+1)}$. Then $p^C = a - bN\frac{a-c}{b(N+1)} = \frac{a+Nc}{N+1}$, and $\pi^C = (p^C - c)q^C = \frac{(a-c)^2}{b(N+1)^2}$.
- It remains to find the deviation profits. In this case, the deviating firm can assume that all other firms are producing the collusion amount, so that $q_{-i} = (N-1)q^m = \frac{N-1}{N}\frac{a-c}{2b} \equiv \bar{Q}$. The deviation quantity is given by $q^D = \arg\max_q \left[a - b(q + \bar{Q}) - c\right] q$. The FOC gives $q^D = \frac{a-c}{2b} - \frac{\bar{Q}}{2} = \frac{a-c}{2b} - \frac{N-1}{N}\frac{a-c}{4b} = \frac{a-c}{4b}\frac{N+1}{N}$. Thus $\pi^D = \left[a - b\left(\frac{a-c}{4b}\frac{N+1}{N} + \frac{N-1}{N}\frac{a-c}{2b}\right) - c\right]\frac{a-c}{4b}\frac{N+1}{N}$. With a bit of algebra we can reduce this to $\pi^D = \frac{(a-c)^2(N+1)^2}{16bN^2}$.

Thus our Nash revision condition becomes

$$\frac{1}{1-\delta}\frac{(a-c)^2}{4bN} \geq \frac{(a-c)^2(N+1)^2}{16bN^2} + \frac{\delta}{1-\delta}\frac{(a-c)^2}{b(N+1)^2}.$$

To solve this, I recommend multiply by $\frac{4bN(1-\delta)}{(a-c)^2}$. Then we get

$$1 \geq (1-\delta)\frac{(N+1)^2}{4N} + \delta\frac{4N}{(N+1)^2}.$$

Let's do a change of variables, setting $x = \frac{(N+1)^2}{4N}$. Then our condition reduces further to $1 \geq (1-\delta)x + \delta\frac{1}{x}$. We can solve this to find that $\delta \geq \frac{x}{1+x}$. Substituting back in, we finally find

$$\delta \geq \frac{\frac{(N+1)^2}{4N}}{1 + \frac{(N+1)^2}{4N}} = \frac{(N+1)^2}{4N + (N+1)^2} \equiv \delta_M.$$

Note that $\delta_M \to 1$ as $N \to \infty$. This means that collusion requires firms to be more and more patient as the number of firms grows. To see why, note that as $N \to \infty$, we have $\pi^M \to 0$ and $\pi^C \to 0$ but $\pi^D \to \frac{(a-c)^2}{16b}$. This means that in the limit, firms get positive profit for deviating and infinitesimal profit for not deviating. So, firms must essentially not discount at all for collusion to be sustained.

Solution 5.

(a) Note that the normal form of the stage game is given by the following matrix:

	a	b
a	$\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$	$(\pi, 0)$
b	$(\pi, 0)$	$\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

Suppose that player firm 1's strategy is to mix with probability δ_1, while firm 2 mixes with probability δ_2.

- Firm 1's min-max payoff is given by

$$\min_{\delta_2}\max_{\delta_1}\left(\delta_1\delta_2\frac{\pi}{2} + \delta_1(1-\delta_2)\pi + (1-\delta_1)\delta_2\pi + (1-\delta_1)(1-\delta_2)\frac{\pi}{2}\right).$$

Note that the main expression can be re-written as $\frac{\pi}{2} + \delta_1(\frac{1}{2} - \delta_2)\pi + \delta_2\frac{\pi}{2}$. First, let δ_2 be given. Then only the middle term of this expression matters for our maximization. In particular, we set $\delta_1 = 1$ if $\delta_2 < \frac{1}{2}$, for a payoff of $\pi - \delta_2\frac{\pi}{2}$, and $\delta_1 = 0$ if $\delta_2 > \frac{1}{2}$, for a payoff of $\frac{\pi}{2} + \delta_2\frac{\pi}{2}$. If $\delta_2 = \frac{1}{2}$ then any $\delta_1 \in [0,1]$ give a payoff of $\frac{3}{4}\pi$.

Taking this maximization under δ_1 into account, we can now reduce the overall problem to

$$\min_{\delta_2} \begin{cases} \pi - \delta_2 \frac{\pi}{2} & \text{if } \delta_2 < \frac{1}{2} \\ \frac{\pi}{2} + \delta_2 \frac{\pi}{2} & \text{if } \delta_2 > \frac{1}{2} \\ \frac{3}{4}\pi & \text{if } \delta_2 - \frac{1}{2}. \end{cases}$$

This problem is solved by $\delta_2 = \frac{1}{2}$. Thus firm 1's min-max payoffs are $\frac{3}{4}\pi$.

- Firm 2's min-max payoffs are given by

$$\min_{\delta_1} \max_{\delta_2} \left(\delta_1 \delta_2 \frac{\pi}{2} + (1 - \delta_1)(1 - \delta_2) \frac{\pi}{2} \right).$$

Again, note this expression can be re-written as $\frac{\pi}{2} + \delta_2 \left(2\delta_1 - 1\right) \frac{\pi}{2} - \delta_1 \frac{\pi}{2}$. If $\delta_1 > \frac{1}{2}$ we set $\delta_2 = 1$ for payoff of $\delta_1 \frac{\pi}{2}$. If $\delta_1 < \frac{1}{2}$ we set $\delta_2 = 0$ for payoff of $\frac{\pi}{2} - \delta_1 \frac{\pi}{2}$. And if $\delta_1 = \frac{1}{2}$ then any $\delta_2 \in [0, 1]$ gives a payoff of $\frac{1}{4}\pi$.

Taking this maximization under δ_2 into account, we now solve

$$\min_{\delta_1} \begin{cases} \delta_1 \frac{\pi}{2} & \text{if } \delta_1 > \frac{1}{2} \\ \frac{\pi}{2} - \delta_1 \frac{\pi}{2} & \text{if } \delta_1 < \frac{1}{2} \\ \frac{1}{4}\pi & \text{if } \delta_1 - \frac{1}{2}. \end{cases}$$

This problem is solved by $\delta_1 = \frac{1}{2}$. Thus firm 2's min-max payoffs are $\frac{1}{4}\pi$.

(b) First, recall that the folk theorem says that in any NE of the repeated game, each player much receive average payoffs at least as great as their min-max payoffs. But since the game is constant-sum, if one firm were to get strictly more than their min-max payoffs, the other firm would necessarily get strictly less. Thus the firms must each get exactly their min-max payoffs in any NE of the infinitely repeated game.