Economics 203: Section 6

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1 Logistics

1.1 Problem Sets 2 and 3 Graded

Problem Sets 2 and 3 have been graded and are available for pick-up in section or lecture. Solutions through Problem Set 4 have been posted on Coursework. I will have Problem Set 4 graded by office hours this Thursday, and I will also post solutions through Problem Set 5 this week.

1.2 Problem Set Feedback

Everyone seems to be doing very well on the problem sets so far. I have a couple things you all can do to make the grading go a bit faster:

- Please attach your problems in order.
- If applicable, it really helps to box your answer. This makes sense for any questions where there is a specific formula or equilibrium that is the desired answer for that question. Obviously you don't need to do it for proof-type questions.

2 Concepts

2.1 Subgame Perfect Nash Equilibrium

By this point in the class we have seen that Nash equilibrium sometimes makes predictions that are not "sensible." In extensive games, this is made especially evident by the fact that Nash equilibrium allows players to make non-credible threats, as in the entry game discussed in lecture. Such self-defeating punishments get by because Nash equilibrium only restricts players to play sensibly on the equilibrium path. If we instead force players to make sensible choices at more points in the game, then we have a more restrictive solution concept that will (hopefully) make more reasonable predictions.

Definition 1. Let t be a node in a dynamic game, $h(t)$ be the info set that contains t, and $S(t)$ be all the successor nodes of t. Then a **proper subgame** of a game is a set of nodes $t \cup S(t)$ along with all mappings from info sets to players, from branches to actions, and from terminal nodes to payoffs, such that

(i) $h(t) = t$ and

(ii) for all $t' \in S(t)$, $h(t') \subseteq S(t)$.

That is, t is a singleton information set, and $S(t)$ (all the nodes following t) contains all the info sets it intersects.

Visually, we can imagining "pruning" the game tree just above the node t. If we can remove this pruned branch without "tearing" any information sets, then we have a valid subgame.

Definition 2. A Nash equilibrium δ^* in behavior strategies is **subgame perfect** iff for every proper subgame, the restriction of δ^* to that subgame is a Nash equilibrium in behavior strategies.

2.2 Finding Subgame Perect Nash Equilibria

First, let's consider finite[1] games of perfect information, where every node is an information set, and so every node is the start of a subgame. Further, let's assume the game is *generic*, meaning that no player is indifferent between any of her payoffs at the terminal nodes. Thus we can use backwards induction to find every SPNE:

- Start at the last non-terminal nodes (i.e. the last nodes where a decision is made). For the player making the choice at each of these nodes, note that if they were to find themselves at this node, they would simply select the highest-payoff option from their available actions.
- Move up the tree one branch. The players acting at these nodes choose the best option by looking at the so-called "continuation" payoffs; that is, they select their highest-payoff option, knowing what their opponents will do after each of their actions by the previous step.
- Continue this process all the way up the game tree.

This should make it clear that a generic, finite game of perfect information has a unique SPNE.

Note, however, that we can use this same approach even if the game in not generic. The only wrinkle is that there may be multiple SPNE, since players may be indifferent between continuation payoffs.

We can even use backwards induction on finite games of *im*perfect information. In these cases, you'll need to imagine subgames that involve non-singleton information sets as games in normal form. Find the (possibly not unique) NE of these games to find the (possibly not unique) continuation payoffs.

Lastly, note that we can use backwards induction for games that have an infinite number of histories but the maximum length of as history is bounded (that is, games of *finite horizon*). In these games, you'll need to be very careful about properly labeling and describing strategies. Suppose, for example, that player 1 moves first in a game with an action $x \in \mathbb{R}$; then there are an infinite number of subgames, one for each possible x. Player 2's strategy needs to describe his actions in every subgame. That is, his strategy will depend on x in general. A strategy profile is then a subgame perfect Nash equilbrium in this game if the specified strategy for player 2 is optimal after *every* x, even if this particular x is never played in equilibrium by player 1.

2.3 Looking Ahead

Note that NE essentially only restricts that "sensible" choices (that is, best responses) be made at the beginning of a game, when players select their strategies. SPNE further restricts that sensible choices be made at any singleton information sets, but leaves open the possibility of strange decisions being made when players have imperfect information. As we will see in lecture later today, perfect Bayesian equilibrium restricts that sensible choices be made at every information set, conditional on beliefs; but these beliefs can be themselves non-sensical if they are off the equilibrium path. Finally, sequential equilibrium restricts that these rationalizing beliefs be consistent, in a sense that we will define in lecture.

3 Problems

Problem 1. *The ultimatum game.*

Consider the following situation, wherein two players split a pie. Player 1 proposes a split, and then player 2 accepts or rejects this split. If player 2 rejects the split, both players get nothing.

(a) Model this situation as a game in extensive form.

(b) What payoffs can be supported by Nash equilibrium?

[1] Remember that by finite, we mean that there is a finite number of possible histories in the game.

(c) Find the subgame perfect Nash equilibria. What payoffs can be supported in this case?

Problem 2. *Five pirates.*

Five pirates have discovered 100 gold coins worth of treasure. They agree to split the treasure by Blackbeard's Rules: The highest-ranked pirate proposes an allocation. All pirates then vote on this allocation simultaneously. If a weak majority approve, the allocation is made; otherwise, the proposer walks the plank and the process is repeated with the next-highest-ranked pirate. Note that pirates enjoy making others walk the plank, though not as much as they like a single gold coin. What is the SPNE outcome of this game?

Problem 3. *Problem Bank #43.*

Nation A plans to attack nation B. The attack can occur at one of two locations, C and D. The success or failure of the attack depends on three factors: where A's troops are massed prior to the attack (near C or near D), where the attack occurs, and which location is defended. Let x denote the fraction of A's troops massed near C, and let $1-x$ denote the fraction of A's troops massed near D. The game is played as follows: simultaneously, A chooses to attack either C or D, and B decides to defend either C or D. Payoffs are determined as follows. If the attack is successful, the payoff to A is 1, and the payoff to B is zero. If the attack is unsuccessful, the payoff to A is zero, and the payoff to B is 1. The probability of a successful attack is in turn determined as follows. Let z denote the fraction of A's troops massed near the location where A attacks (so $z = x$ if A attacks at C, and $z = 1 - x$ if A attacks at D). Then the probability of a successful attack is z if B does not defend the location where A attacks, and $\frac{z}{2}$ if B does defend the location where A attacks.

(a) Depict the normal form of this game. Are there ranges of x for which either or both players have dominant strategies? If so, indicate which strategies are dominant over which ranges. Are there ranges of x for which pure strategy Nash equilibria exist? If so, indicate the ranges, and characterize the Nash equilibria (specify equilibrium strategies and indicate payoffs). Are there ranges of x for which mixed strategy Nash equilbria exist? If so, indicate the ranges and characterize the mixed strategy Nash equilibria (specify equilibrium strategies and indicate payoffs). Draw a graph representing A's expected equilibrium payoff as a function of x.

(b) Now imagine that, instead of being fixed, x (the deployment of troops) is under A's control. Events occur in the following order: first, A decides how many troops to mass near each location (that is, A chooses x); next, B observes the deployment of A's troops (that is, B observes x); finally, A chooses to attack either C or D, and simultaneously B decides to defend either C or D. Solve for the subgame perfect equilibria of this game. How does A deploy its troops?

4 Solutions

Solution 1.

(a) Let the pie be of size 1. Then player 1's strategy is to choose $x \in [0, 1]$. Player 2's strategy is a function $f : [0, 1] \to \{Accept, Reject\}$. Note that player 2's strategy gives his accept/reject decision for every possible split of the pie. If 2 accepts, payoffs are $(x, 1 - x)$. If he rejects, payoffs are $(0, 0)$.

(b) Consider the following strategy profile: Player 1 picks $x = k$ and player 2 accepts iff $x = k$, for some $k \in [0, 1]$. Note that player 1 is getting $k \geq 0$, whereas any deviation will get him payoff 0, so he has no incentive to deviate. Player 2 also has no incentive to deviate, since no strategy can get him more than $1 - k$, given that 1 is playing $x = k$. Thus we have a NE. So, we see that any payoffs of the form $(k, 1 - k)$ for $k \in [0, 1]$ can be supported by a NE.

Note also that the strategy profile where 1 plays $x = 1$ and 2 always rejects is also a NE. Both players get 0 in equilibrium, and can do no better by deviating. Thus the payoffs $(0, 0)$ can also be supported by a NE.

(c) Consider a subgame where 1 has just played $x = k$. If $k > 0$, 2's unique best response is to accept. Thus any SPNE strategy profile must have $f(k) = \textit{Accept}$ for $k > 0$. If $k = 0$, then 2 is indifferent between accepting and rejecting, so either is a best response.

Now consider player 1's choice at the beginning of the game. Any $x = k > 0$ will get him payoff $k > 0$. Thus the strategy profile where $x = 1$ and 2 always accepts is a SPNE. The only other possible SPNE is one where 2 rejects iff $x = 0$, but in this case 1's best response does not exists, so we can't have a SPNE. Thus in the only SPNE of this game, the payoffs are $(1, 0)$.

Solution 2.

Let's call the 5 pirates A, B, C, D, and E, ranked in that order. The key to this game is to note that while there are millions of subgames, they can be grouped according to who is making the decision at the beginning of that subgame. At any subgame where a particular pirate is making his allocation decision, all the pirates ranked above him are dead and all the coins remain to be allocated. Whatever proposals were made before have no effect on the subgame.

First, consider the subgame where pirate E proposes an allocation. He clearly allocates everything to himself, since he is the only surviving pirate.

Next, consider the subgame where pirate D proposes an allocation. He knows that he can win any vote by a tie, so he allocates everything to himself and votes for this allocation. Pirate E's vote has no effect on the outcome in this case, so he can vote up or down. The payoffs are 100 for D and 0 for E.

Next, consider the subgame where pirate C proposes an allocation. Note that if his proposal is not passed, he will walk the plank, and the continuation payoffs will be 100 for D and 0 for E. Note that by allocating at least 1 coin to E, C can guarantee that his proposal will pass, since he and E will vote for it.[2] Thus in this subgame the payoffs are 99 for C, 0 for D, and 1 for E.

Next, the subgame where pirate B proposes. By the same logic as above, he notes that he must get at least one other pirate to vote up his proposal, and the cheapest way to do this is to give 1 coin to pirate D, who would otherwise get 0 in continuation. Thus the payoffs are 99 for B, 0 for C, 1 for D, and 0 for E.

Finally, we arrive at the top of the game tree. Note that 1 needs the support of two other pirates to avoid walking the plank. But noting that C and E get 0 in continuation if A's proposal is rejected, A can offer just 1 coin each to C and E to get their support. Thus the SPNE outcome is that A gets 98 coins, B gets 0, C gets 1, D gets 0, and E gets 1.

Solution 3.

(a) First, we need to construct the normal form, which is given in Figure 1. Note that since payoffs are 1 if a nation is successful and 0 if not, then payoffs for a particular strategy profile are just the probability of success for each nation. Note also that since x is fixed for this part, and nations attack/defend simultaneously, each nation has just 2 strategies: attack/defend C and attack/defend D.

	C	D
C	$(\frac{x}{2}, 1 - \frac{x}{2})$	$(x, 1 - x)$
D	$(1 - x, x)$	$(\frac{1-x}{2}, \frac{1+x}{2})$

Figure 1: The normal form of the game in part (a). Nation A chooses rows and Nation B chooses columns.

The answers to this part are summarized in Table 1.

[2]Here the tiebreaking rule is important: If he were to get 0 coins either way, E would rather make D walk the plank.

x	Dominant strat.	PSNE	MSNE	NE payoffs
$< 1/3$	D for A	(D, D)	none	$\left(\frac{1-x}{2}, \frac{1+x}{2}\right)$
$= 1/3$	none	(D, D)	$\left\{((p, 1-p), D) : p \leq \frac{2}{3}\right\}$	$(1/3, 2/3)$
$\in (1/3, 2/3)$	none	none	$((1-x, x), (3x-1, 2-3x))$	$\left(\frac{3x(x-1)}{2}, 1 - \frac{3x(x-1)}{2}\right)$
$= 2/3$	none	(C, C)	$\left\{((p, 1-p), C) : p \geq \frac{1}{3}\right\}$	$(1/3, 2/3)$
$> 2/3$	C for A	(C, C)	none	$\left(\frac{x}{2}, 1 - \frac{x}{2}\right)$

Table 1: Summary for part (a).

To see the dominant strategies results, note that for C to be dominant for player A, we must have $\frac{x}{2} > 1 - x$ and $x > \frac{1-x}{2}$. These conditions are jointly satisfied iff $x > \frac{2}{3}$. By a similar method we can show that D is dominant for A iff $x < \frac{1}{3}$. For C to be dominant for player B, we need $1 - \frac{x}{2} > 1 - x$ and $x > \frac{1+x}{2}$. But these conditions cannot both be true for $x \in [0, 1]$, so C can never be dominant for B. Similarly, we can show that D can never be dominant for B as well.

Next we look for PSNE. Note that if $x < 1/3$ the game is dominance solvable and the unique NE is (D, D). Similarly, the unique NE for $x > 2/3$ is (C, C). Payoffs are $(\frac{1-x}{2}, \frac{1+x}{2})$ and $(\frac{x}{2}, 1 - \frac{x}{2})$, respectively. One can also verify directly that for $x = 1/3$, (D, D) is a PSNE (with payoffs $(1/3, 2/3)$), and for $x = 2/3$, (C, C) is a PSNE (with payoffs $(1/3, 2/3)$). Lastly, for the range $x \in (1/3, 2/3)$, one can show that there are no PSNE, since $\frac{x}{2} > 1 - x$, $x < \frac{1+x}{2}$, $1 - \frac{x}{2} > 1 - x$, and $x < \frac{1+x}{2}$ in that range.

Finally, we need to look for MSNE where applicable. Consider the case $x = 1/3$. One can show easily that a 2-by-2 MSNE is not possible. The only possibility for a 2-by-1 MSNE is that player B plays C and A mixes with weight p on C. A is clearly best-responding, since he get payoff $1/3$ from C and from D. Then for B to be best-responding, we must have $p \geq 1/3$. Thus we have the family of MSNE $\left\{((p, 1-p), D) : p \leq \frac{2}{3}\right\}$, where payoffs are $(1/3, 2/3)$ for all these MSNE. Similarly, we can show that for $x = 2/3$, we have the family of MSNE $\left\{((p, 1-p), C) : p \geq \frac{1}{3}\right\}$, where again payoffs are $(1/3, 2/3)$ for all these MSNE. And lastly, we consider the case $x \in (1/3, 2/3)$. Note that 2-by-1 MSNE are not possible, since all best-responses are strict. Thus, direct calculation by the usual methods gets the MSNE $((1-x, x), (3x-1, 2-3x))$, with payoffs $\left(\frac{3x(x-1)}{2}, 1 - \frac{3x(x-1)}{2}\right)$.

The graph of A's payoffs as a function of x is given in Figure 2.

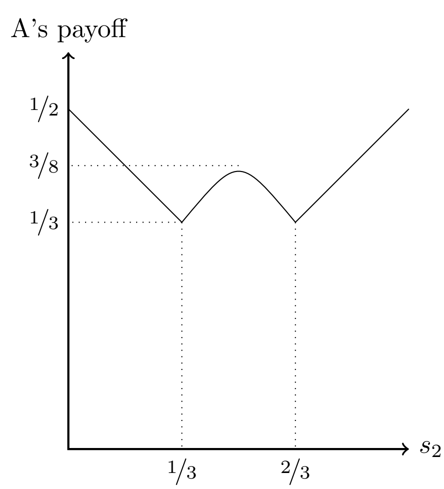

Figure 2: Nation A's NE payoff as a function of x.

(b) Note that the game has one proper subgame for each possble x, but we've solved for the NE in all of these games in the previous part. Thus nation A simply chooses the continuation game to play (i.e. the x) that gives it the highest NE payoff in the corresponding subgame. As our graph in the previous part makes clear, nation A can choose either $x = 1$ or $x = 0$ for a payoff of $\frac{1}{2}$. That is, A masses all of its troops at one location, and then attacks that location.