Economics 203: Section 4

Jeffrey Naecker

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1 Logistics

1.1 Reminder: Problem Set 4 Due Date Extended

Please note that the deadline for Problem Set 4 has been extended. The problem set is now due Tuesday, February 14, in section. It can also be turned into my box by 9:00 a.m. that day.

1.2 Solutions Posted

I've posted solutions for Problem Bank problems 1 through 14. This should cover all the problems for the first three problem sets.

1.3 Problem Set 1 Graded

Problem Set 1 has been graded and will be returned today. Overall everyone did very well on the problem set. I noticed few issues that tripped some of you up, and I had some general comments:

- It can't hurt to show your work on the problem set. This is a good habit for exams, where you can get a lot of partial credit even if your final answer is wrong.
- In problem 3, you gave a formula for the number of strategies a player has as a function of the number of information sets and actions they have. Use this formula to check your work! On problems 1, 3, and 4, some of you did not have the right number of strategies even though you had drawn the extensive forms correctly.
- In general, you don't need to worry about nature as a player when asked list strategy profiles or otherwise describe strategies.
- When nature moves in a game, the numbers in each entry of a normal form game should be the expected payoffs for each player, given the corresponding strategies. The expectation should be over nature's possible moves. This gave some of you trouble on Problem 2, so please double-check your answers against the solutions.

2 Concepts

2.1 Defining a Mixed Strategy Nash Equilibrium

As Doug noted in lecture, Pure Strategy Nash equilibria often do not exist. Often in these settings, randomization of strategies seems like a reasonable response. We formalize this intuition with a new solution concept: mixed strategy Nash equilibrium (MSNE), which we define in the following way.

Consider the normal form game $\{\{S_i\}_{i=1}^I, g\}$. Let $K_i = |S_i| < \infty$ enumerate player i's pure strategies. Then we can let Δ_i be the K_i-dimensional simplex, so that $\delta_i \in \Delta_i$ is the probability vector over i's pure strategies; that is, $\delta_{ik} = P(i \text{ plays pure strategy } s_i^k)$. We call a particular δ_i a **mixed strategy** of player i.

We then define $\Delta = \times_{i \in I} \Delta_i$ as the mixed strategy profile set, and $\delta \in \Delta$ as a mixed strategy profile. Then the function $\pi : \Delta \to \mathbb{R}^I$ assigns payoffs for each $\delta \in \Delta$ as follows:

$$\pi_i(\delta) = E_S[g_i(s)|\delta] = \sum_{s \in S} g_i(s) P(s|\delta).$$

That is, each δ gives a probability distribution over all possible pure strategy profiles, each of which corresponds to a payoff for player i. That player's payoff for δ is then just the expectation of these payoffs given by the implied probability distribution.

Using these definition, we now define a new game $\{\{\Delta_i\}_{i=1}^I, \pi\}$ from the original game $\{\{S_i\}_{i=1}^I, g\}$.

Definition 1. A **mixed strategy Nash equilibrium** of the game $\{\{S_i\}_{i=1}^I, g\}$ is a pure strategy Nash equilibrium of the game $\{\{\Delta_i\}_{i=1}^I, \pi\}$.

In the above definition, we've defined MSNE as a PSNE of a different game. We can also define a MSNE more directly:

Definition 2. A **mixed strategy Nash equilibrium** of the game $\{\{S_i\}_{i=1}^I, g\}$ is a mixed strategy profile $\delta^* \in \Delta$ such that

$$E_S[g_i(s)|\delta_i^*, \delta_{-i}^*] \geq E_S[g_i(s)|\delta_i, \delta_{-i}^*] \quad \forall \delta_i \neq \delta_i^* \quad \forall i \in I.$$

2.2 Finding a Mixed Strategy Nash Equilibrium

Finding the mixed strategy Nash equilibria of a game is made fairly straightforward by the following observation:

Theorem 1. *In a MSNE, each player must be indifferent between all pure strategies to which her mixed strategy attaches positive probability.*

Proof. Suppose that the mixed strategy profile δ^* is a MSNE. Note that we can write player i's payoffs in in the following way:

$$\begin{aligned} E_S[g_i(s)|\delta_i^*, \delta_{-i}^*] &= E_S[g_i(s)|\delta_i^*, \delta_{-i}^*] \\ &= E_{S_i}[E_{S_{-i}}[g_i(s_i, s_{-i})|\delta_{-i}^*]|\delta_i^*] \\ &= \sum_{k=1}^{K_i} \delta_{ik}^* E_{S_{-i}}[g_i(s_i^k, s_{-i})|\delta_{-i}^*]. \end{aligned}$$

Suppose without loss of generality that i is playing s_i^1 and s_i^2 in this MSNE, and that $E_{S_{-i}}[g_i(s_i^1, s_{-i})|\delta_{-i}^*] > E_{S_{-i}}[g_i(s_2^k, s_{-i})|\delta_{-i}^*]$. That is, given the other players are playing according to δ^*, player i is *not* indifferent between s_i^1 and s_i^2. But then we have the following:

$$\begin{aligned} E_S[g_i(s)|\delta_i^*, \delta_{-i}^*] &= \delta_{i1}^* E_{S_{-i}}[g_i(s_i^1, s_{-i})|\delta_{-i}^*] + \delta_{i2}^* E_{S_{-i}}[g_i(s_i^2, s_{-i})|\delta_{-i}^*] + \ldots + \delta_{iK_i}^* E_{S_{-i}}[g_i(s_i^{K_i}, s_{-i})|\delta_{-i}^*] \\ &< \delta_{i1}^* E_{S_{-i}}[g_i(s_i^1, s_{-i})|\delta_{-i}^*] + \delta_{i2}^* E_{S_{-i}}[g_i(s_i^1, s_{-i})|\delta_{-i}^*] + \ldots + \delta_{iK_i}^* E_{S_{-i}}[g_i(s_i^{K_i}, s_{-i})|\delta_{-i}^*] \\ &= (\delta_{i1}^* + \delta_{i2}^*) E_{S_{-i}}[g_i(s_i^1, s_{-i})|\delta_{-i}^*] + \ldots + \delta_{iK_i}^* E_{S_{-i}}[g_i(s_i^{K_i}, s_{-i})|\delta_{-i}^*] \\ &= E_S[g_i(s)|\delta_i, \delta_{-i}^*] \end{aligned}$$

for some $\delta_i \neq \delta_i^*$. Thus δ_i^* is not a best response, and so δ^* is not a MSNE. □

Note an important implication of this result: It is your indifference condition that ties down your opponent's probabilities in a MSNE. We see this at work in the following simple example.

	B	S
B	$(2,1)$	$(0,0)$
S	$(0,0)$	$(1,2)$

Figure 1: The normal form of the game Bach or Stravinsky.

Example 1. Consider the game Bach or Stravinsky, whose normal form is given in Figure 1. We can see that the game has two PSNE: (B, B) and (S, S).

Next, let us consider possible MSNE. We can parameterize each player's mixed strategies as follows: Let $p = P(s_1 = B)$ and $q = P(s_2 = B)$. Then a mixed strategy profile is of the form $((p, 1-p), (q, 1-q))$.

It is important to note that there can be no MSNE where 1 player mixes but the other player plays a pure strategy. This is because given a player's pure strategy, his opponent has a unique best response. Therefore, we look for MSNE where both players are strictly mixing.

Given player 2's mixing, player 1 must be indifferent between playing B and playing S:

$$\begin{aligned} E[g_1(B)|q] &= E[g_1(S)|q] \\ 2q + 0(1-q) &= 0q + 1(1-q) \\ q &= \frac{1}{3} \end{aligned}$$

And given player 1's mixing, player 2 must be indifferent between playing B and playing S:

$$\begin{aligned} E[g_2(B)|q] &= E[g_2(S)|p] \\ p + 0(1-p) &= 0p + 2(1-q) \\ p &= \frac{2}{3} \end{aligned}$$

Thus $\left((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\right)$ is a MSNE.

For the visually inclined, it can be very helpful to draw the best-responses for both players in $p-q$ space. It is straightforward to check that 1's best-response to q is as follows:

$$p = BR_1(q) = \begin{cases} 0 & \text{if } 0 \leq q < \frac{1}{3} \\ \in [0,1] & \text{if } q = \frac{1}{3} \\ 1 & \text{if } \frac{1}{3} < q \leq 1. \end{cases}$$

Note that as we showed above, when $q = \frac{1}{3}$, 1 is indifferent between his two pure strategies, and thus between any mixture of the two as well. Similarly, we can find 2's best response:

$$q = BR_1(p) = \begin{cases} 0 & \text{if } 0 \leq p < \frac{2}{3} \\ \in [0,1] & \text{if } p = \frac{2}{3} \\ 1 & \text{if } \frac{2}{3} < p \leq 1. \end{cases}$$

If we graph these two best response curves, we can see that all the MSNE are given by the intersections of the two curves. Of course, a PSNE is a special case of MSNE.

The above methods work very well for finite games where each player has only a few strategies. We need to employ some different techniques when the strategy sets become continuous, as the following example shows.

Example 2. Consider the Betrand model with 2 firms and capacity constraints, as in Doug's lecture notes. Let's assume without proof that a symmetric MSNE exists, where each player chooses their price according to a probability distribution whose support is $[a, v]$. Let F be the CDF of this distribution. Let's also assume

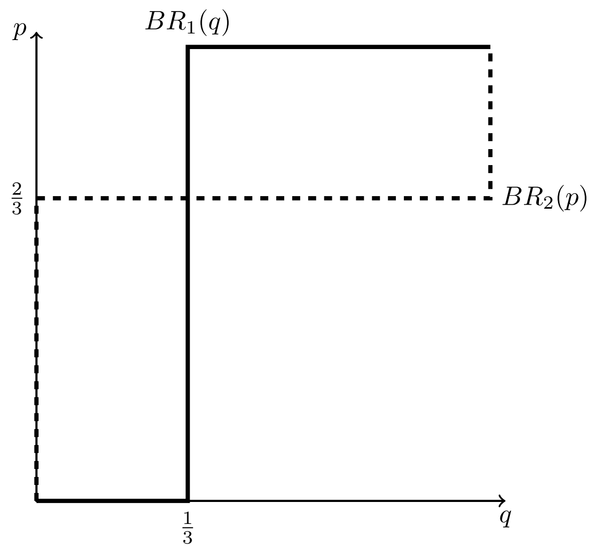

Figure 2: Seeing a the MSNE of Bach or Stravinsky.

that the CDF is atomless; this tells us there will be no ties (or more accurately, the event that the two firms naming the same price has zero probability of occurring).

Given these assumptions, let's find the CDF of this MSNE. To do so, imagine yourself as one of the firms, and assume your opponent is naming a price according to F. You are naming a price p in the support of F. If your opponent's price is higher than p, you sell K units for profit Kp. Note that this happens with probability $1 - F(p)$. Similarly, if your opponent's price is lower than p, you sell $(Q-K)$ units for a profit of $(Q-K)p$. This happens with probability $F(p)$. Thus your expected profit for naming p when your opponent is playing according to F is

$$(Q-K)pF(p) + Kp\,(1 - F(p)) = C,$$

where the constant C reminds us that for any p, your expected payoff must be the same. We can then solve for F:

$$F(p) = \frac{C - Kp}{(Q-2K)p}.$$

Note that we have two more terms to nail down, however: the constant C and the lower bound a. We will find the two with two tricks that will be very useful for these types of problems.

First, we note that $F(v) = 1$ by definition. This allows us to find that $C = (Q-K)v$. Note that this can be interpreted as your profits as a firm in this game playing the MSNE. This makes a lot of sense, because if you deviate to $p = v$, you will sell only $Q-K$ units, since the other firm is undercutting you with probability 1; thus your profit is just $(Q-K)v$. (Note that this is effectively another way to find C.) Thus form of the CDF is completely nailed down:

$$F(p) = \frac{K}{2K-Q} - \frac{(Q-K)v}{(2K-Q)p}.$$

Lastly, we note that $F(a) = 0$ by definition. This allows us to find $a = \left(\frac{Q}{K} - 1\right)v$. Note that $a > 0$ since $K < Q$, so all named prices are positive. Note also that $a < v$ since $Q < 2K$, so the distribution is not degenerate.

3 Problems

Problem 1. *A game in extensive form.*

Consider the game shown in Figure 3.

(a) What is the normal form of this game?

(b) Find all MSNE of this game.

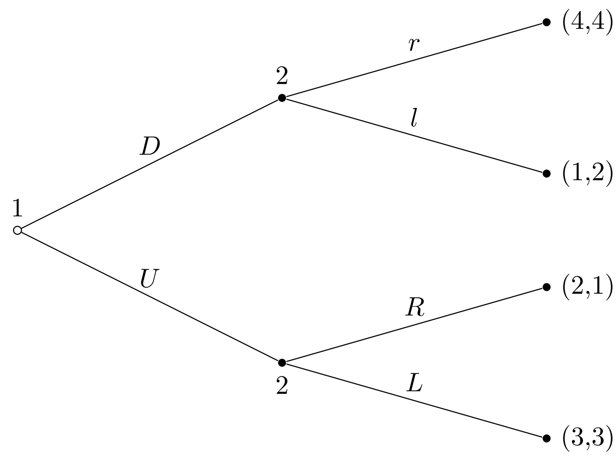

Figure 3: The game for problem 1.

Problem 2. *Asymmetric all-pay auction (Problem Bank #22).*

Consider an asymmetric all-pay auction between two bidders whose values are common knowledge. The object is worth v to Bidder 1 and v' to Bidder 2 where $v' > v$. All bidders simultaneously submit sealed bids b_i which must be non-negative; the highest bidder wins the object and all bidders pay their bids. In the event of a tie, the good is given to Bidder 2.

(a) What bidding strategies are strictly dominated? Can you iteratively eliminate any more strictly dominated strategies?

(b) Are there any pure-strategy Nash equilibria? If so, characterize the set of PSNE. If not, explain why not.

We now consider a mixed-strategy Nash equilibrium where both bidders randomize.

(c) At a MSNE, would Bidder 1 ever bid strictly higher than v with positive probability? Would Bidder 2 ever bid strictly higher than v with positive probability?

(d) Consider a MSNE where Bidder i randomly selects a bid based on the CDF F_i. You can assume (without proof) that the support of F_i is $[0, v]$.

   (i) Write an expression for 2's payoff from bidding x in $[0, v]$, fixing F_1.

   (ii) Use the fact that $x = v$ is in the support of Bidder 2's strategy to solve for 2's payoff.

   (ii) Use answers from the last two parts to solve for $F_1(x)$. What is $F_1(0)$?

   (iv) Write an expression for 1's payoff from bidding x in $[0, v]$, fixing F_2. Use the fact that $x = v$ is in the support of Bidder 1's strategy to solve for 1's payoff.

   (v) Use answers from the previous part to solve for $F_2(x)$. What is $F_2(0)$?

(e) What is the expected revenue? As v' varies in the set (v, ∞), is expected revenue increasing, constant, or decreasing in v'?

4 Solutions

Solution 1.

(a) The normal form is given in Figure 4.

	Ll	Lr	Rl	Rr
U	$(3,3)$	$(3,3)$	$(2,1)$	$(2,1)$
D	$(1,2)$	$(4,4)$	$(1,2)$	$(4,4)$

Figure 4: The normal form of the game in problem 1.

(b) We start by looking for any strategies that are removed by IDDS, as these will not be played with positive probability in a MSNE.[1] We can see that Rl is dominated by Lr for player 1, and unfortunately this is as far as we can get with IDDS. You can confirm this by showing that all other strategies are rationalizable. (Remember the set of rationalizable strategies is the same as the set of strategies surviving IDDS for two-player games.)

Next, you can easily check that there are 3 PSNE: (U, Ll), (D, Lr), and (D, Rr).

Lastly, it remains to find the MSNE. Note that if 1 is putting positive probability on U and D, then 2's best response is to play Lr only. To see this, note that if 1 is putting probability p on U, then 2's payoff from Lr is strictly higher than her payoff from Ll or Rr: $3p + 4(1-p) > 3p + 2(1-p)$ and $3p + 4(1-p) > 1p + 4(1-p)$. But if 2 is playing only Lr, then 1's unique best-response is to play D only. This is a contradiction, so no MSNE can involve 1 strictly mixing. Thus we just need to check two case.

Case 1: 1 plays U only. In this case, 2 is indifferent between playing Ll and playing Lr, so any mixture of the two is a best response for her. By that mixture must be such that 1 playing U is a best response as well. So, we need the payoff from U to be at least as big as the payoff from D, given that 2 is putting probability q on playing Ll. That is,

$$3q + 3(1-q) \geq 1q + 4(1-q),$$

which reduces to $q \geq \frac{1}{3}$. Thus $((1,0)(q, 1-q, 0, 0))$ is a MSNE for any $q \geq \frac{1}{3}$.

Case 2: 1 plays D only. In this case, 2 is indifferent between playing Lr and playing Rr, so any mixture of the two is a best response for her. By the same logic as above, we need the payoff from D to be at least as big as the payoff from U, given that 2 is putting probability q on playing Lr. That is,

$$4q + 4(1-q) \geq 3q + 2(1-q),$$

which is satisfied for any valid q. Thus $((0,1)(0, q, 0, 1-q))$ is a MSNE for any $q \in [0,1]$.

Note that the PSNE we found above are included in the two families of MSNE.

Solution 2.

(a) For player i, bidding $b_i > v_i$ gives a negative payoff for sure, but bidding $b_i = 0$ guarantees a non-negative payoff. Thus $b_1 > v$ and $b_2 > v'$ are dominated.

Given that 1 will not bid more than v, $b_2 > v$ is dominated by $b_2 = v$. (Note the tiebreaking rule here.) Thus we eliminate $b_2 > v$ with one round of iterative deletion.

[1] This is not mentioned in the lecture notes, but you can use it without proof in problem sets and on the exam.

Now, given that both bidders are bidding in $[0, v]$, can we do any further rounds of deletion? It turns out no. Given $b_1 \in [0, v]$, 2's best response is to set $b_2 = b_1$. Thus all of 2's strategies are best responses for some strategy of 1, and can't be deleted.

Showing that none of 1's remaining strategies is dominated is more difficult. We can't use the same argument as above, since 1's best response does not exist. Instead, we show that any $b_i \in [0, v]$ can't be dominated. Suppose, by way of contradiction, that some $\widehat{b}_1 \in [0, v]$ is dominated by a pure strategy. That is, there exists some $\tilde{b}_1$ such that $g_1(\tilde{b}_1, b_2) > g_1(\widehat{b}_1, b_2)$ for all $b_2 \in [0, v]$. But the consider two cases:

- Consider $\tilde{b}_1 < \widehat{b}_1$. Suppose that $b_2 \in (\tilde{b}_1, \widehat{b}_1)$. Then $g_1(\tilde{b}_1, b_2) = -\tilde{b}_1 \leq 0 \leq v - \widehat{b}_1 = g_1(\widehat{b}_1, b_2)$, a contradiction.
- Consider $\tilde{b}_1 > \widehat{b}_1$. Suppose that $b_2 > \tilde{b}_1$. Then $g_1(\tilde{b}_1, b_2) = -\tilde{b}_1 < -\widehat{b}_1 = g_1(\widehat{b}_1, b_2)$, a contradiction.

So, we can't have $b_1 \in [0, v]$ dominated by a pure strategy. One can show by similar methods and a lot more algebra that the same is true for a mixed strategy. Thus no $b_1 \in [0, v]$ is dominated.

(b) There are no PSNE. We proceed by cases:

- Suppose $0 < b_i < b_j$. Then j could lower his bid slightly but still win the auction, thus improving payoffs.
- Suppose $0 < b_1 = b_2$. In this case, 1 will lose the auction, giving him a negative payoff. He can do better by bidding 0 for a payoff of 0.
- Suppose $0 = b_1 = b_2$. Then player 1 can deviate to a bid greater than 0 to win the object and guarantee positive payoffs.

(c) We know that any strategy that is eliminated by IDDS will not have positive weight in a MSNE. Thus neither bidder will bid above v with positive probabillty.

(d) (i) If 2 bids $b_2 = x$, she wins the object with probability $F_1(x)$ for a payoff of $v' - x$, and loses with probabillty $1 - F_1(x)$ for a payoff of $-x$. Thus 2's expected payoff from bidding $b_2 = x$ is $(v' - x)F_1(x) + (-x)(1 - F_1(x)) = v'F_1(x) - x$.

(ii) If 2 bids v, she wins the object for sure and gets payoff $v' - v$. This must be her payoff in the MSNE, since she must get the same payoff from each pure strategy in her support.

(ii) We know 2's payoffs are $v'F_1(x) - x = v' - v$. Solving for $F_1(x)$, we find that

$$F_1(x) = \frac{v' - v + x}{v'}.$$

Note that $F_1(0) = \frac{v'-v}{v'}$.

(iv) If 1 bids x, he wins the object with probability $F_2(x)$ for a payoff of $v - x$, and loses with probabililty $1 - F_2(x)$ for a payoff of $-x$. Thus 1's expected payoff from bidding $b_2 = x$ is $(v - x)F_2(x) + (-x)(1 - F_2(x)) = vF_2(x) - x = C$. Since $F(v) = 1$, we find that $C = 0$. Alternatively, note that if 1 bids v he gets payoff 0 with probability 1.

(v) Thus we find that

$$F_2(x) = \frac{x}{v},$$

noting that $F_2(0) = 0$.

(e) Note The expected revenue from player 1 is $\int_0^v x \frac{1}{v'} dx = \frac{v^2}{2v'}$, and the expected revenue from player 2 is $\int_0^v x \frac{1}{v} dx = \frac{v}{2}$. Thus total expected revenue is $\frac{v^2}{2v'} + \frac{v}{2}$, which is decreasing in v'.