Economics 203: Section 3

Jeffrey Naecker

January 31, 2012

1 Logistics

1.1 Reminder: Problem Set 3 Modified

Please note that I updated Problem Set 3. You need to do only problems 11 and 14 from the problem bank. The problem set is still due at 5:00 pm in my box on Friday, February 3.

2 Concepts

2.1 Pure Strategy Nash Equilibrium

Last week's lectures focused exclusively on applications of Pure Strategy Nash Equilbirum:

Definition 1. A strategy profile $s^* = (s_1^*, \dots, s_I^*)$ is a **pure strategy Nash equilibrium (PSNE)** iff

$$g_i(s_i^*, s_{-i}^*) \geq g_i(s_i, s_{-i}^*)$$

for all $s_i \in S_i$ and for all $i \in I$.

2.1.1 Finding PSNE

Often it is useful to write down the **best-response correspondence** of a player i given the other players' strategies,

$$BR_i(s_{-i}) = \arg\max_{s_i \in S_i} g_i(s_i, s_{-i}).$$

We can then write an overall best-response correspondence as $BR(s) = (BR_1(s_{-1}), \dots, BR_I(s_{-I}))$. A PSNE s^* is a fixed point of this correspondence: $s^* = BR(s^*)$. Solving this equation for s^* will give us the PSNE strategy profiles.

For example, consider the simple case of a Cournot setting (quantity competition) where firms have identical linear costs (i.e. $c_i(q_i) = cq_i$) and demand is linear: $p = a - bQ$. Let's assume we have just 2 firms, and consider firm 1's best response function:

$$BR_1(q_2) = \arg\max_{q_1} \left[a - b(q_1 + q_2)\right] q_1 - cq_1.$$

The first-order condition tells us that $BR_1(q_2) = \frac{a-c}{2b} - \frac{q_2}{2}$. We can see that firm 2's problem is identical, so that firm's best response is $BR_2(q_1) = \frac{a-c}{2b} - \frac{q_1}{2}$. A PSNE is a fixed point of the overall best-response function; that is, it is a solution to the following system of equations:

$$q_1 = \frac{a-c}{2b} - \frac{q_2}{2}$$
$$q_2 = \frac{a-c}{2b} - \frac{q_1}{2}$$

Solving this system of equations[1] yields the PSNE $q_1^* = q_2^* = \frac{a-c}{3b}$.

We can also use the best-response correspondences to help us find PSNE visually. If we have 2 players in a game, we can create a graph with player 1's strategies on one axis and player 2's strategies on the other. We then plot each player's best response(s) as a function of their opponent's strategies. Any point where the best responses overlap or intersect is a PSNE. This is because at these points, both players are mutually best responding.

We can see this approach in action with the Cournot example in Figure 1.

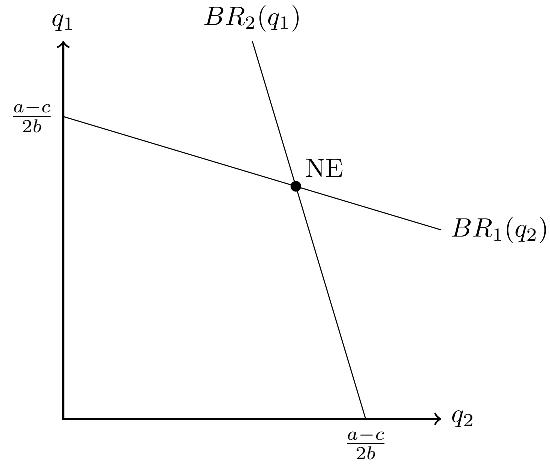

Figure 1: Seeing a PSNE graphically in the Cournot example.

3 Problems

Problem 1. *The savings game (Problem Bank #13).*

Consider a weekly work-consumption-savings plan of a household consisting of two individuals. Each of the individuals, $i = 1, 2$, decides on his/her own work level w_i and level of consumption of a frivolous good f_i. Assume that $w_i \in [0, 1600]$ and $f_i \in [0, 5000]$. Given their work and consumption decisions, the amount of money the household can put in the bank (the household's savings) is given by $b = (w_1 + w_2) - (f_1 + f_2)$.

(a) Suppose that both members of the household have the same utility function which is given by

$$u(w_1, w_2, f_1, f_2) = 2\min\{b, 500\} - (w_1 + w_2) + 0.1(f_1 + f_2)$$

(i) Demonstrate that all (w_i, f_i) with $w_i > 0$ and $f_i > 0$ are strictly dominated for player i. What other strategies, if any, are strictly dominated for player i?

(ii) Which pure strategies survive iterated elimination of strictly dominated strategies?

(iii) Identify the pure strategy Nash equilibria of this game.

(b) Continue to assume that the payoff of player 1 is as above, but that the payoff of player 2 is instead

$$u_2(w_1, w_2, f_1, f_2) = 2\min\{b, 500 + \varepsilon\} - (w_1 + w_2) + 0.1(f_1 + f_2)$$

for $\varepsilon > 0$.

[1]The symmetry of the problem allows us to take an algebraic shortcut: Since the firms' problems are identical, a symmetric PSNE must exist. That allows us to look simply for q^* such that $q^* = \frac{a-c}{2b} - \frac{q^*}{2}$.

(i) What is the set of pure strategy Nash equilibria? What is the set of strategies for each player that survives iterated elimination of strictly dominated strategies?

(ii) Explain what happens in the limit as ε approaches 0. Consider the correspondence which for each ε gives the set of Nash equilibria. Is this an upper hemicontinuous correspondence?

Problem 2. *War of attrition (Osbourne & Rubinstein 18.5)*

Two players are involved in the dispute over an object. The value of the object to player i is $v_i > 0$. Time is a continuous variable that starts and 0 and runs indefinitely. Each player chooses when to concede the object to the other player. If the first player to concede does so at time t, the other player receives the object at that time. If both players concede at the same time, the object is split equally between them, player i receiving a payoff of $\frac{v_i}{2}$. Time is valuable: until the first concession both players lose one unit of payoff per unit time.

(a) Formulate this situation to a a strategic game. What are the strategy sets of the players? What are the payoff functions?

(b) Write down and draw the best-response correspondences for each player.

(c) Show that in all PSNE of this game, one player concedes immediately.

Problem 3. *NE and dominance solvability (Problem Bank #8).*

Consider a finite game, and suppose that it is dominance solvable. Prove that it has a unique Nash equilibrium and that this coincides with the iterated dominance solution.

4 Solutions

Solution 1.

(a) First, we show that (w_i, f_i) where $w_i > 0$ and $f_i > 0$ is dominated. To do this, notice that if we decrease both w_i and f_i by a little bit, we leave b unchanged, and the rest of the utility function actually increases. Formally, consider $(\tilde{w_i}, \tilde{f_i})$ such that $\tilde{w_i} = w_i - \varepsilon$ and $\tilde{f_i} = f_i - \varepsilon$. But then note that

$$\begin{aligned} u_i(\tilde{w_i}, \tilde{f_i}, w_j, f_j) &= 2\min\{\tilde{w_i} + w_j - \tilde{f_i} - f_j, 500\} - (\tilde{w_i} + w_j) + \frac{1}{10}(\tilde{f_i} + f_j) \\ &= 2\min\{w_i - \varepsilon + w_j - f_i + \varepsilon - f_j, 500\} - (w_i + w_j) + \varepsilon + \frac{1}{10}(f_i + f_j) - \frac{1}{10}\varepsilon \\ &= u_i(w_i, f_i, w_j, f_j) + \frac{9}{10}\varepsilon \\ &> u_i(w_i, f_i, w_j, f_j). \end{aligned}$$

Thus any $(w_i, f_i) >> 0$ is dominated by $(\tilde{w_i}, \tilde{f_i})$ such that $\tilde{w_i} = w_i - \varepsilon$ and $\tilde{f_i} = f_i - \varepsilon$ for some $\varepsilon > 0$.

Before proceeding with the rest of the problem, it is now useful to write down and draw the best response functions for the players. In general, this would be really hard, as each player's strategies are 2-dimensional, so drawing a best-response would require 4 dimensions. But the previous part tells us that in fact each player's strategy will be either of the form $(w_i, 0)$ or of the form $(0, f_i)$, both of which are one-dimensional! This allows us to draw the best responses fairly easily.

Suppose j is playing $(w_j, 0)$. What is i's best response? He could play $(w_i, 0)$ for some $w_i \geq 0$, in which case his payoffs would be as follows:

$$u_i(w_i, w_j) = 2\min\{w_i + w_j, 500\} - (w_i + w_j) = \begin{cases} w_i + w_j & \text{if } w_i + w_j \leq 500 \\ 1000 - (w_i + w_j) & \text{if } w_i + w_j \geq 500 \end{cases}$$

Note that in the first case i wants to increase w_i, whereas in the second case, he wants to decrease w_i. This means that for any w_j, the optimal $w_i = 500 - w_j$, or as close to that as possible. Thus for $w_j \leq 500$, $w_i = w_j - 500$ for a payoff of 500, whereas for $w_j \geq 500$, $w_i = 0$ for a payoff of $1000 - w_j \leq 500$.

But this is not quite yet i's best response function, because i could also play $(0, f_i)$ for some $f_i \geq 0$, in which case his payoffs would be as follows:

$$u_i(f_i, w_j) = 2\min\{w_j - f_i, 500\} - w_j + \frac{1}{10} f_j = \begin{cases} w_j - \frac{19}{10} f_i & \text{if } w_j - f_i \leq 500 \\ 1000 - w_j + \frac{1}{10} f_i & \text{if } w_j - f_i \geq 500 \end{cases}$$

By similar logic as above, the optimal $f_i = w_j - 500$, or as close to that as possible. Thus for $w_j \leq 500$, i should set $f_i = 0$ for a payoff of w_j, whereas for $w_j \geq 500$, he should set $f_i = w_j - 500$ for a payoff of $950 - \frac{9}{10} w_j \geq 1000 - w_j$.

Now we can fully describe i's best response function, given that j is playing $(w_j, 0)$:

$$BR_i(w_j) = \begin{cases} (500 - w_j, 0) & \text{if } w_j \leq 500 \\ (0, w_j - 500) & \text{if } w_j \geq 500 \end{cases}$$

Next, suppose that j is instead playing $(0, f_j)$. If i chooses to play $(w_i, 0)$, his payoff will be as follows:

$$u_i(w_i, f_j) = 2\min\{w_i - f_j, 500\} - w_i + \frac{9}{10} f_j = \begin{cases} w_i - \frac{19}{10} f_j & \text{if } w_i - f_j \leq 500 \\ 1000 - w_i + \frac{1}{10} f_j & \text{if } w_i - f_j \geq 500 \end{cases}$$

By the logic we've already seen a couple times, if i is playing $(w_i, 0)$, his best option is play $w_i = 500 + f_j$ for a payoff of $500 - \frac{9}{10} f_j$ if $f_j \leq 1100$. He should play $w_i = 1600$ for a payoff of $1600 - \frac{19}{10} f_j$ if $f_j \geq 1100$.

If i instead responds by playing $(0, f_i)$, his utility is

$$\begin{aligned} u_i(f_i, f_j) &= 2\min\{-f_i - f_j, 500\} + \frac{1}{10} f_i + \frac{1}{10} f_j \\ &= -\frac{19}{10} f_i - \frac{19}{10} f_j. \end{aligned}$$

Clearly in this case i wants to minimize f_i. So, if i is playing $(0, f_i)$, his best option is $f_i = 0$ for a payoff of $-\frac{19}{10} f_j$.

Now we can write down i's best response to $(0, f_j)$:

$$BR_i(f_j) = \begin{cases} (500 + f_j, 0) & \text{if } f_j \leq 1100 \\ (1600, 0) & \text{if } f_j \geq 1100 \end{cases}$$

We can now immediately see that the following strategies for i are not dominated, since they are best responses:

- $(w_i, 0)$ for any $w_i \in [0, 500]$ (best response to $w_j = 500 - w_i$),
- $(w_i, 0)$ for any $w_i \in [500, 1600]$ (best response to $f_j = w_i - 500$), and
- $(0, f_i)$ for any $f_i \in [0, 1100]$ (best response to $w_j = f_i + 500$).

The only strategies remaining to be considered are $(0, f_i)$ for $f_i > 1100$. The work we've done for far suggests that any such strategy can be improved upon by lowering f_i. And in fact, $(0, 1100)$ dominates $(0, f_i)$ for $f_i > 1100$:

$$\begin{aligned} u_i(0, f_i > 1100) &= 2\min\{-f_i + w_j - f_j, 500\} - w_j + \frac{1}{10} f_i + \frac{1}{10} f_j \\ &= -2f_i + 2w_j - 2f_j - w_j + \frac{1}{10} f_i + \frac{1}{10} f_j \\ &< -2 * 1100 + 2w_j - 2f_j - w_j + \frac{1}{10} 1100 + \frac{1}{10} f_j \\ &= u_i(0, 1100) \end{aligned}$$

Thus $(0, f_i)$ for $f_i > 1100$ is dominated.

If we draw the best response functions, we can also see that we have the following PSNE:

- $((w_i, 0), (w_j, 0))$ for any w_i, w_j such that $w_i + w_j = 500$, and
- $((w_i, 0), (0, f_j))$ where $w_i \in [500, 1600]$ and $f_j = w_i - 500$.

We can also verify that these are PSNE by inspecting the best response functions we've written down. In the first NE, for example, i is best responding to j because $w_j \leq 500$ and $w_i = 500 - w_j$ as dictated by i's best response. The situation for j is symmetric, so she is best responding as well.

Finally[2], it is clear that the NE strategies are exactly the strategies remaining after IDDS, since all other strategies are dominated. More precisely, the strategies surviving IDDS are

- $(w_i, 0)$ for any $w_i \in [0, 1600]$ and
- $(0, f_i)$ for any $f_i \in [0, 1100]$

for $i = 1, 2$.

(b) We are going to show that there is in fact a unique strategy profile that survives IDDS, and thus a unique PSNE. The general logic is that the the best response functions will tell us which strategies might be dominated (those that are not a best response), and then we will directly prove that these strategies are in fact dominated.

It is useful to note that player 1's best-response function is the one we constructed in the previous part. For player 2, we replace 500 with $500 + \varepsilon$, so we have that her best-response function is given by the following if 1 plays $(w_1, 0)$:

$$BR_2(w_1) = \begin{cases} (500 + \varepsilon - w_1, 0) & \text{if } w_1 \leq 500 + \varepsilon \\ (0, w_1 - 500 - \varepsilon) & \text{if } w_1 \geq 500 + \varepsilon \end{cases}$$

If 1 instead plays $(0, f_1)$, then 2's best response is as follows:

$$BR_2(f_1) = \begin{cases} (500 + \varepsilon + f_1, 0) & \text{if } f_1 \leq 1100 - \varepsilon \\ (1600, 0) & \text{if } f_1 \geq 1100 - \varepsilon \end{cases}$$

We start by eliminating dominated strategies. For both players, this eliminates $(w_i, f_i) >> 0$ as in the previous part. For player 1, $(0, f_1 > 1100)$ is also eliminated, also as above, since nothing changes in the arguments in the previous part. For player 2, if we follow the logic in the previous part, we can show that $(0, f_2 > 1100 - \varepsilon)$ is dominated. Thus the strategies remaining are as follows:

$$\begin{array}{ll} (w_1, 0) \text{ s.t. } w_1 \in [0, 1600] & (w_2, 0) \text{ s.t. } w_2 \in [0, 1600] \\ (0, f_1) \text{ s.t. } f_1 \in [0, 1100] & (0, f_2) \text{ s.t. } f_1 \in [0, 1100 - \varepsilon] \end{array}$$

Next, note that we've eliminated part of the support of 1's best response function. Now, $(w_1 > 1600 - \varepsilon, 0)$ is no longer a best response to anything. And in fact, it is dominated by $(1600 - \varepsilon, 0)$, as the following argument shows:

$$\begin{aligned} u_1(w_1 > 1600 - \varepsilon, 0) &= 2\min\{\underbrace{w_1}_{>1600-\varepsilon} + \underbrace{w_2 - f_2}_{>-1100+\varepsilon}, 500\} - w_1 - w_2 + \frac{1}{10}f_2 \\ &= 1000 - w_1 - w_2 + \frac{1}{10}f_2 \\ &< 1000 - (1600 - \varepsilon) - w_2 + \frac{1}{10}f_2 \\ &= u_1(1600 - \varepsilon, 0) \end{aligned}$$

[2]Note that I am not answering the questions in the same order they are asked. Because of the relationship between solution concepts, this is often an effective test strategy.

So, we eliminate $(w_1 > 1600 - \varepsilon, 0)$.

But now we have eliminated part of the support of 2's best-response function, so that $(0, f_2 > 1100-2\varepsilon)$ is no longer a best-response. We can show that $(0, f_2 > 1100 - 2\varepsilon)$ is dominated by $(0, 1100 - 2\varepsilon)$:

$$\begin{aligned} u_2(0, f_2 > 1100 - 2\varepsilon) &= 2\min\{\underbrace{w_1 - f_1}_{<1600-\varepsilon} + \underbrace{-f_2}_{<-1100+2\varepsilon}, 500 + \varepsilon\} - w_1 + \frac{1}{10}f_1 + \frac{1}{10}f_2 \\ &= 2w_1 - 2f_1 - 2f_2 - w_1 - w_2 + \frac{1}{10}f_2 \\ &< 2w_1 - 2f_1 - 2(1100 - 2\varepsilon) - w_1 - w_2 + \frac{1}{10}(1100 - 2\varepsilon) \\ &= u_2(0, 1100 - 2\varepsilon) \end{aligned}$$

So, we eliminate $(0, f_2 > 1100-2\varepsilon)$. But note that we have lopped off another chunk of the support of 1's best-response, so we can eliminate $(w_1 > 1600 - 2\varepsilon, 0)$. Then we can eliminate $(0, f_2 > 1100 - 3\varepsilon)$, and so on. This continues until we hit a bound, namely $f_2 = 0$. At this point, we have eliminated $(w_1 > 500 - \varepsilon, 0)$ and $(0, f_2 > 0)$. Thus the strategies remaining are as follows:

$$\begin{array}{ll} (w_1, 0) \text{ s.t. } w_1 \in [0, 500 - \varepsilon] & (w_2, 0) \text{ s.t. } w_2 \in [0, 1600] \\ (0, f_1) \text{ s.t. } f_1 \in [0, 1100] & (0, 0) \end{array}$$

Now we see that with $w_1 > 500 - \varepsilon$ eliminated, $(w_2 < \varepsilon, 0)$ is no longer a best response for player 2. The usual arguments show that in fact $(w_2 < \varepsilon, 0)$ is dominated by $(w_2 = \varepsilon, 0)$, so we eliminate $(w_2 < \varepsilon, 0)$. But then $(w_1 > 500 - \varepsilon, 0)$ is dominated, so we eliminate it. We continue this process until we hit a bound, $w_1 = 0$. At that point, we have eliminated $(w_1 > 0, 0)$ and $(w_2 < 500 + \varepsilon, 0)$. Thus the strategies remaining are as follows:

$$\begin{array}{ll} (0, 0) & (w_2, 0) \text{ s.t. } w_2 \in [500 + \varepsilon, 1600] \\ (0, f_1) \text{ s.t. } f_1 \in [0, 1100] & -- \end{array}$$

Lastly, note that we've now removed the part of 1's best-response that supported playing $(0, f_1 > \varepsilon)$. We can show that strategies of this form are dominated by $(0, f_1 = \varepsilon)$, and so are eliminated. But then we can show that $(w_2 < 500 + 2\varepsilon, 0)$ is dominated, and so is eliminated. Continuing this process until we run into a bound $(w_2 = 1600)$, we eliminate all $(0, f_1 < 1100)$ and all $(w_2 < 1600, 0)$. The remaining strategies are then just as $(0, 1100)$ for 1 and $(1600, 0)$ for 2.

Thus $((0, 1100), (1600, 0))$ is the unique NE and the unique profile that survives IDDS.

We are asked to consider the correspondence from ε to PSNE profiles. Note that for $\varepsilon > 0$, the set of PSNE profiles is the singleton $((0, 1100), (1600, 0))$. When $\varepsilon = 0$, the set of PSNE is given in part (a). But $((0, 1100), (1600, 0))$ is in this set, so the correspondence is upper hemicontinuous.

Solution 2.

(a) The players's stategies are to name $t_i \in [0, \infty)$. The payoff function is as follows:

$$g_i(t_i, t_j) = \begin{cases} -t_i & \text{if } t_i < t_j \\ \frac{v_i}{2} - t_i & \text{if } t_i = t_j \\ v_i - t_j & \text{if } t_i > t_j \end{cases}$$

(b) If my opponent is waiting longer than my valuation of the object, I don't want to hold out to win, since this will guarantee a negative payoff. I even don't want to try to tie my opponent, since this also guarantees a negative payoff. My best response is in fact to concede immediately. The same logic holds if my opponent is waiting exactly my valuation. Lastly, if my opponent gives up before my valuation, I can guarantee myself positive payoffs by waiting just longer than him. Thus my best response function is as follows:

$$BR_i(t_j) = \begin{cases} 0 & \text{if } t_j \geq v_i \\ t \in (t_2, \infty) & \text{if } t_j < v_i \end{cases}$$

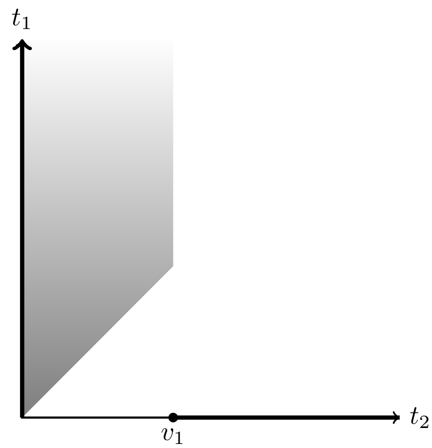

Figure 2: The shaded area and thick lines indicate the best response correspondence for player 1.

(c) By drawing the best response functions of each player on the same graph, it should be clear that we have two types of PSNE: $(t_1 = 0, t_2 \geq v_1)$ and $(t_1 \geq v_2, t_2 = 0)$. In both of these types of PSNE, one player concedes immediately.

Solution 3.

Let s^* be the dominance solvability outcome.

- First, we show that s^* is a PSNE.

  Suppose not. Then there exists $\hat{s}_i$ such that $g_i(\hat{s}_i, s^*_{-i}) > g_i(s^*_i, s^*_{-i})$. Now, we know that $\hat{s}_i$ was eliminated by IDDS, and let's say that it was eliminated after k rounds of deletion. Let S^k be the strategy profiles remaining just before $\hat{s}_i$ was eliminated. Thus there exists some ρ such that

$$\sum_{s_i^m \in S_i^k} \rho_m g_i(s_i^m, s_{-i}) > g_i(\hat{s}_i, s_{-i}) \quad \forall s_{-i} \in S^k_{-i}.$$

  But note that $s^*_{-i} \in S^K_{-i}$ since s^*_{-i} survives IDDS, so in fact the above equation implies

$$\sum_{s_i^m \in S_i^k} \rho_m g_i(s_i^m, s^*_{-i}) > g_i(\hat{s}_i, s^*_{-i}).$$

  But $g_i(\hat{s}_i, s^*_{-i}) > g_i(s^*_i, s^*_{-i})$, so in fact we have that

$$\sum_{s_i^m \in S_i^k} \rho_m g_i(s_i^m, s^*_{-i}) > g_i(s^*_i, s^*_{-i}).$$

  Thus once all elements of S_{-i} except s^*_{-i} are deleted, s^*_i is dominated and should be deleted as well! This is a contradiction, so it must be that s^* is a PSNE.

- Next, we show that any $\hat{s} \neq s^*$ is not a PSNE.

  Suppose $\hat{s} \neq s^*$ is a PSNE. Note that $\hat{s}$ was eliminated by IDDS at some point, so assume without loss of generality that $\hat{s}_i$ is eliminated before $\hat{s}_{-i}$, and let this happen in round k of IDDS. Then we have that

  $$\sum_{s_i^m \in S_i^k} \rho_m g_i(s_i^m, s_{-i}) > g_i(\hat{s}_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}^k,$$

  where S^k is defined as before. Since $\hat{s}_{-i} \in S_{-i}^k$, the above implies that

  $$\sum_{s_i^m \in S_i^k} \rho_m g_i(s_i^m, \hat{s}_{-i}) > g_i(\hat{s}_i, \hat{s}_{-i}).$$

  But for this to be true, there must be some $\tilde{s}_i \neq \hat{s}_i$ such that $g_i(\tilde{s}_i, \hat{s}_{-i}) > g_i(\hat{s}_i, \hat{s}_{-i})$. This is a contradiction to $\hat{s}$ being a PSNE profile. Thus $\hat{s}$ is not a PSNE profile.