Economics 203: Section 2

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1 Logistics

1.1 Reminder: Problem Sets

Remember, problem sets are generally due no later than 5:00 p.m. on Friday. They should be turned in only to my box, on the second floor of the Landau building near the entrance to the lounge. I won't accept copies handed in to me or emailed to me, so please plan accordingly.

1.2 Reminder: Office Hours

Time: Thursday 1:30 p.m. - 3:30 p.m.
Location: Landau 245

2 Concepts

2.1 Revisiting (Strict) Dominance

2.1.1 A checklist

How do you quickly tell whether a strategy s_i is or is not dominated? Here is a checklist:

1. Is there a pure strategy $\hat{s_i} \neq s_i$ that dominates s_i? If so, then s_i is dominated.
2. Is s_i the (not necessarily unique) best response to some s_{-i}? If so, then s_i is *not* dominated.
3. Is there a mixture of other strategies for i that dominates s_i? If so, then s_i is dominated.

The last item is just the definition of s_i being a dominated strategy. My point here is simply that you don't have to immediately start looking for some complicated mixture of strategies to tell whether or not a strategy is dominated. The first two items on the checklist may give you a very quick answer.

2.1.2 Dealing with mixed strategies

But what happens if you get all the way to item 3 on the checklist, and a dominating mixture doesn't jump out at you? One strategy is to start writing down constraints on ρ, the vector of probability weights, that the definition of dominance requires.

Example 1. Consider the normal form game in Figure 1, which we saw last week. The checklist quickly establishes that for player 2, a and b are not dominated (since they are best responses to player 1 choosing A and B, respectively). But items 1 and 2 of the checklist fail to resolve the status of c. So, we need to find a mixture of a and b that dominates c, or prove that one does not exist.

	a	b	c
A	$(4, 10)$	$(3, 0)$	$(1, 3)$
B	$(0, 0)$	$(2, 10)$	$(10, 3)$

Figure 1: A normal form game.

Note that the definition of a strictly dominated strategy tells us that we are looking for ρ_a and ρ_b such that

$$\rho_a \cdot 10 + \rho_b \cdot 0 > 3 \qquad \text{and}$$
$$\rho_a \cdot 0 + \rho_b \cdot 10 > 3,$$

which imply that $\rho_a > \frac{3}{10}$ and $\rho_b > \frac{3}{10}$. From there, it is easy to pick a combination of ρ_a and ρ_b that dominates c.

Suppose instead that we were facing the game given in Figure 2. Again a and b are not dominated. This time, however, we find that a dominating mixture requires $\rho_a > \frac{6}{10}$ and $\rho_b > \frac{6}{10}$. But this contradicts $\rho_a + \rho_b = 1$. Thus in this game c is not dominated.

	a	b	c
A	$(4, 10)$	$(3, 0)$	$(1, 6)$
B	$(0, 0)$	$(2, 10)$	$(10, 6)$

Figure 2: A normal form game with payoffs in the last column modified.

2.2 Weak Dominance

We see many examples where strict dominance does not rule out strategies that seem fairly silly to us. (As in the Less-Than game, for example.) The reason for this is that strict dominance is actually fairly demanding: for a strategy to be strictly dominated, there must be another strategy out there that always does *strictly* better. Well, what if we relax this a bit, and look for strategies that do at least as well all the time, but only strictly better some of the time? What we get in this case is weak dominance.

Definition 1. A strategy s_i^k is a **weakly dominated strategy** for player i iff there exists a measure ρ on S_i such that

$$\sum_{m=1}^{K_i} \rho_m g_i(s_i^m, s_{-i}) \geq g_i(s_i^k, s_{-i})$$

for all $s_{-i} \in S_{-i}$, with at least one inequality strict.

When applying weak dominance, we don't iterate as with strict dominance, since the order of deletion matters. Thus our "solution concept" in this case is just to list strategies that are or are not weakly dominated. Hopefully this gets us a unique solution, but in many cases there is a multiplicity of profiles that consist of strategies that are not weakly dominated.

2.2.1 Another checklist

Just as with strict dominance, we can apply a checklist to quickly find whether or not a strategy is weakly dominated:

1. Is there a pure strategy $\hat{s}_i \neq s_i$ that weakly dominates s_i? If so, then s_i is dominated.

2. Is s_i the *unique* best response to some s_{-i}? If so, then s_i is *not* dominated.

3. Is there a mixture of other strategies for i that weakly dominates s_i? If so, then s_i is dominated.

Again, the point here this that you don't need to immediately start looking for a mixture that weakly dominates. Most strategies can be quickly categorized by items 1 and 2 above.

2.2.2 Dealing with many cases

When the size of the strategy set becomes large or infinite, you will often need to check several cases. I find it helpful to keep track of things in the following way.

Suppose I'm pretty sure that strategy s_i^l weakly dominates strategy s_i^k. Suppose further that I can partition the strategy set(s) of the other player(s) in some useful way, say into three segments A, B, C such that $A \cup B \cup C = S_{-i}$. Then I can make something like the following table:

	s_i^k		s_i^l
$s_{-i} \in A$	$g_i(s_i^k, s_{-i})$	$<$	$g_i(s_i^l, s_{-i})$
$s_{-i} \in B$	$g_i(s_i^k, s_{-i})$	$\leq$	$g_i(s_i^l, s_{-i})$
$s_{-i} \in C$	$g_i(s_i^k, s_{-i})$	$=$	$g_i(s_i^l, s_{-i})$

As long as there are no "greater than" or "greater than or equal to" symbols in there, and at least one strict "less than", then indeed s_i^l weakly dominates s_i^k.

Example 2. Recall the sealed-bid second-price auction for the lecture notes. Let p_i be i's bid and p_{-i}^m be the maximum bid among all other players. Suppose i chooses to play some $p_i = p' < v_i$. We can show this is weakly dominated by playing $p_i = p'' = v_i$:

	$p_i = p' < v_i$		$p_i = p'' = v_i$
$p_{-i}^m < p'$	$v_i - p_{-i}^m$	$=$	$v_i - p_{-i}^m$
$p_{-i}^m = p'$	$\frac{1}{n}\left(v_i - p_{-i}^m\right)$	$<$	$v_i - p_{-i}^m$
$p_{-i}^m \in (p', p'')$	0	$<$	$v_i - p_{-i}^m$
$p_{-i}^m \geq p''$	0	$=$	0

A similar table shows that bidding more that your valuation is also weakly dominated.

2.3 Rationalizability

Definition 2. (i) A strategy $s_i \in S_i$ is **1-rationalizable** if it is a best response to some (independent) probability distribution over strategies of all other players.

(ii) A strategy $s_i \in S_i$ is **k-rationalizable** if it is a best response to some (independent) probability distribution over $(k-1)$-rationalizable strategies of all other players.

(iii) A strategy is **rationalizable** if it is k-rationalizable for all k.

Note that in weak dominance, we don't require a player to take a stand on what his opponent is actually doing; instead, we ask what strategies are best for any possible strategy of this opponent. But this may not get us very far, so what if we require a player to consider what is going on in the mind of his opponent? And similarly, his opponent should be considering what he is doing, and so on. This logic leads us to rationalizability.

This may sound a lot like iterated deletion of dominated strategies to you, and indeed the two solution concepts are closely related. As we saw in lecture, the set of strategies that are rationalizable is a subset of the set of strategies that survive IDDS. In fact, with two players, these sets are identical.

Note. In section we had an extensive discussion on how to show that a strategy is rationalizable. One approach is to draw best-response "chains" as on page 28 the lecture notes. Several of you had a question about the interpretation of mixed strategies when drawing these figures. The important thing to keep straight is that a strategy is k-rationalizable if is a best response to some mixture of $(k-1)$-rationalizable strategies. This is NOT saying that the mixture itself be $(k-1)$-rationalizable, but that each strategy in the mixture is *individually* $(k-1)$-rationalizable. This is a subtle but important distinction.

For a concrete example, consider the game on page 28 of the lecture notes. We can interpret the "chain" figures as saying that d is a best response to a mixture of A and B, each of which are rationalizble, so d is rationalizable. We are not saying that the mixture $\frac{1}{2}A + \frac{1}{2}B$ is rationalizable.

2.4 Nash Equilibrium

In all of our solution concepts so far, we have not required a player's beliefs about her opponent's actions to be correct. Nash equilibrium, however, captures a steady state where players respond rationally to their expectations, and these expectations are in fact borne out in the equilibrium.

Definition 3. A strategy profile $s^* = (s_1^*, \dots, s_I^*)$ is a **pure strategy Nash equilibrium (PSNE)** iff

$$g_i(s_i^*, s_{-i}^*) \geq g_i(s_i, s_{-i}^*)$$

for all $s_i \in S_i$ and for all $i \in I$.

It is immediately obvious that all PSNE strategies are rationalizable (since all players are best-responding), and thus survive IDDS.

2.4.1 Finding PSNE

You have almost certainly seen the standard approach to finding PSNE in a normal form game. For each of player 2's strategies, underline the payoff of player 2 corresponding to their best response to that strategy. Then do the same with the roles of players reversed. Any strategy profile (i.e. cell) with both payoffs underlined will be a PSNE. See, for example, the Prisoner's Dilemma in Figure 3.

	NF	F
NF	$(-2, -2)$	$(-5, \underline{-1})$
F	$(\underline{-1}, -5)$	$(\underline{-4}, \underline{-4})$

Figure 3: The Prisoner's Dilemma. (F, F) is the unique PSNE.

If we are dealing with a game with an infinite number of strategies, we'll need a different approach. Often it is useful to write down the **best-response correspondence** of a player i given the other players' strategies, $BR_i(s_{-i})$. We can then write an overall best-response correspondence as $BR(s) = (BR_1(s_{-1}), \dots, BR_I(s_{-I}))$. A PSNE is then a fixed point of this correspondence: $s^* = BR(s^*)$. Solving this equation for s^* will then give us the PSNE strategy profiles.

3 Application: Games in the Real World

At this point, we've already seen a lot of fairly sophisticated solution concepts. But do these solution concepts predict in any way how people plays games in the real world? Let's look at some important examples.

3.1 Chess

As mentioned in the lecture notes, chess is finite game of perfect information, so Zermelo's theorem clearly says that a PSNE exists for this game. Chess has only three possible outcomes: White wins, Black wins, or a draw. This means that exactly one of the following statements must be true:

(i) In every PSNE of chess, White wins.

(ii) In every PSNE of chess, Black wins.

(iii) In every PSNE of chess, the game ends in a draw.

Yet we observe all three outcomes all the time, even when the same players compete. Thus players are clearly not playing the PSNE strategies. This is not surprising, of course, given the complexity of the game. Chess has approximately 10^{47} legal positions, and is believed to have about 10^{123} possible games that can be played[1]. By comparison, the age of the universe is believed to be about 10^{17} seconds, so if you played one game of chess per second, it would take you 10^{106} universe-ages to play all games of chess. Since an average board has 35 legal moves for each player, even thinking just 5 rounds ahead would require almost one universe-age of computation at that rate.

3.2 The Centipede Game

What about a game that requires a much more reasonable amount of backwards induction? Consider the centipede game introduced in the notes; surely an average human can calculate the Nash equilibrium of that game. Yet a recent paper found that in a 6-period version of the game, only 12% of game ended in the first period[2]. If complexity is not an issue, what else could explain this outcome? Perhaps players are rational, and capable of the appropriate calculations, but not sure that their opponents are rational. This lack of common knowledge of rationality can then support any outcome of the game.

3.3 The Dictator Game

Lastly, suppose we remove any issues of rationality from the strategic considerations of the players. To do so, we have the Dictator Game. In this game, player 1 names an amount $x \in [0, 10]$. The payoffs are then $(10 - x, x)$. Player 2 makes no moves, so this is really just a one-person decision problem, with player 1's optimal strategy being $x = 0$. Yet in the lab, this is not what we observe at all. We see many players choose $x > 0$, often with a local max at $x = 5$. See, for example, the results from Fisman, Kariv, and Markovits (2007) in Figure 4.

Clearly this game is not too complex, and clearly common knowledge of rationality is not a concern. But there are many factors that explain this behavior, and they all have to do with the fact that in the lab, we can't really assign vNM utility values to outcomes. Much can be going on in the mapping from dollars to utility; for example, players might have preferences for fairness or equity. The dollar values also fail to capture other dimensions of utility: perhaps players care about how they are perceived by the experimenter of the other players, for example.

4 Problems

Problem 1.

Consider the normal-form game given in Figure 5.

(a) Show that all strategies are rationalizable.

[1] This last number is the so-called Shannon number. See `http://en.wikipedia.org/wiki/Game_complexity` for more information.

[2] See Mcintosh, Shogren, and Moravec, (2009).

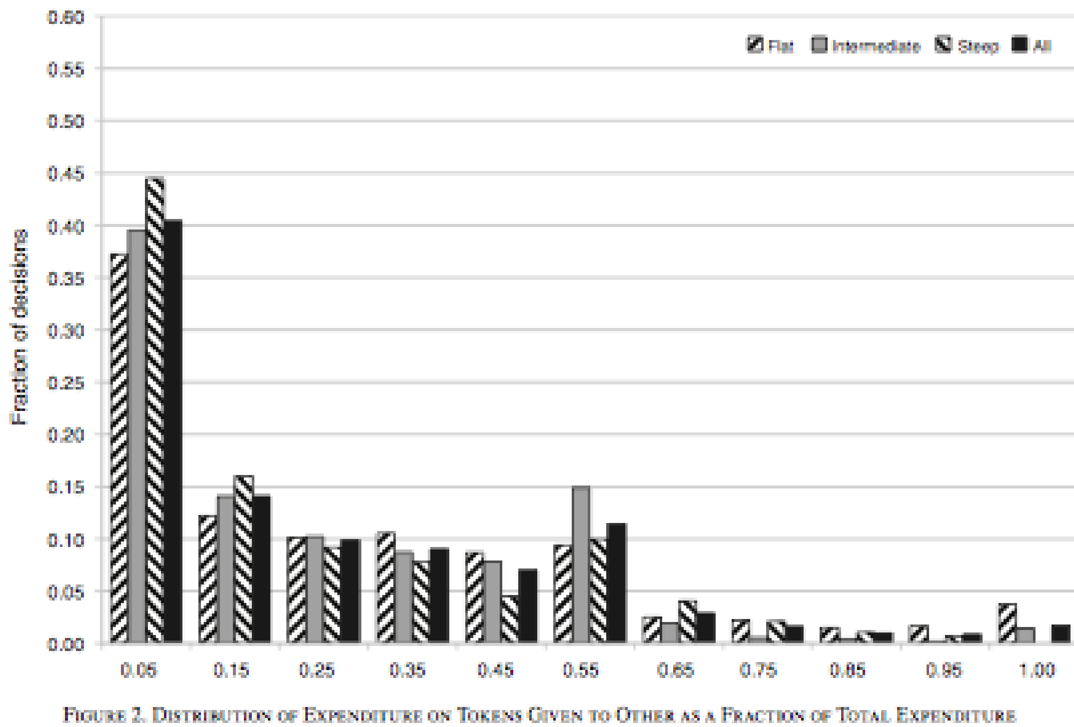

Figure 4: Distribution of dictator game outcomes from Fisman, Kariv, and Markovits (2007).

(b) What are the PSNE for the game?

	l	m	r
U	$(1,3)$	$(0,0)$	$(0,2)$
D	$(0,0)$	$(1,3)$	$(0,2)$

Figure 5: The normal form game for problem 1.

Problem 2. *Garbage-ville.*[3]

Consider a village with $N > 2$ inhabitants. Each inhabitant can choose either litter or be tidy. If k inhabitants litter, then every inhabitant incurs a disutility ka. Further, the litterers incur disutility from social stigma $\frac{S}{k}$. (You can think of there being a fixed amount of stigma in the village and all litterers share this burden equally.) If a villager is tidy, they incur a disutility from effort equal to t. Lastly, let us assume that $1 < \frac{S}{t-a} - 1 < N - 1$.

(a) What is the utility for a litterer when there are k total litterers? What is the utility for a tidier in this case?

(b) Let k_{-i} be the number of litterers not including i. What is i's best correspondence as a function of k_{-i}?

(c) Characterize all of the PSNE of this game. (Hint: Proceed by cases, paying attention to edge cases and jumps in the best response correspondence.)

[3] Based on a problem by Maciej Kotowski.

Problem 3. *The centipede game (Problem Bank #7).*

In class, we constructed a pure strategy Nash equilibrium of the centipede game in which player 1 says "stop" at the initial node. Prove that player 1 says "stop" at the root node in every pure strategy Nash equilibrium.

5 Solutions

Solution 1.

(a) U is a best reponse to l, which is in turn a best response to U, so both U and l are rationalizable. Similarly, D and m are mutual best responses, and thus rationalizable as well. Lastly, r is a best response to $\frac{1}{2}U + \frac{1}{2}D$, and both U and D are rationalizable, so r is rationalizable.

(b) The only pure-strategy NE are (U, l) and (D, m).

Solution 2.

(a) Let's call i's possible strategies L_i for littering and T_i for tidying. Then we have the following:

$$u_i(L_i, k) = -\frac{S}{k} - ka \qquad \text{and}$$
$$u_i(T_i, k) = -t - ka,$$

where recall k is the total number of litterers.

(b) Let's consider i's decision of whether or not to litter when there are k_{-i} other litterers. L_i is a best response iff

$$\begin{aligned} u_i(L_i, k_{-i} + 1) &\geq u_i(T_i, k_{-i}) \\ -\frac{S}{k_{-i} + 1} - (k_{-i} + 1)a &\geq -t - k_{-i}a \\ k_{-i} &\geq \frac{S}{t - a} - 1, \end{aligned}$$

where I've omitted a few steps of algebra. Thus the best response correspondence is

$$BR(k_{-i}) = \begin{cases} L_i & \text{if } k_{-i} \geq \frac{S}{t-a} - 1 \\ T_i & \text{if } k_{-i} \leq \frac{S}{t-a} - 1. \end{cases}$$

(c) We are going to proceed to check by cases. Note that since players are interchangeable, any outcome can be described simply by k, the number of litterers. Note also that in general we need to confirm that both the litterers and the tidiers are best-responding in a given outcome for that outcome to be an equilbrium. If either of these types is not best-responding, then we do not have an equilibrium.

- Is $k = 0$ (i.e. everybody tidies) an equilibrium? Well, if $k = 0$ then $k_{-i} = 0$ as well. But $\frac{S}{t-a} - 1 > 0$ by assumption, so T_i is the best response for player i. Thus all players are best-responding, so $(T_1, T_2, \ldots, T_N)$ is a PSNE.
- Is $1 \leq k < \frac{S}{t-a}$ an equilibrium? For the litterers, $k_{-i} < \frac{S}{t-a} - 1$, so they are not best-responding, and would want to deviate to T_i. Thus we do not have an equilbrium.
- Is $k = \frac{S}{t-a}$ a PSNE? Note that for the tidiers, $k_i = k = \frac{S}{t-a} > \frac{S}{t-a} - 1$, so their best response is in fact to litter. Thus we do not have an equilibrium.

- Is $\frac{S}{t-a} < k < N$ an PSNE? Note again that for the tidiers, $k_{-i} = k > \frac{S}{t-a} > \frac{S}{t-a} - 1$, so they prefer to deviate to T_i. So, not a PSNE.
- Is $k = N$ a PSNE? Everyone is littering in this case, so for all agents $k_{-i} = N - 1 > \frac{S}{t-a} - 1$ by assumption. Thus all agents are best-responding, and so $(L_1, L_2, \ldots, L_N)$ is a PSNE.

In summary, the only PSNE are $(T_1, T_2, \ldots, T_N)$ and $(L_1, L_2, \ldots, L_N)$.

Solution 3.

First, some notation. Let's say a strategy for a player i just lists their choice of either stop (S) or continue (C) at every one of their information sets. So, for example, we could have $s_i = (CCSSCSCC\ldots)$.

Let's consider some strategy profile s^* where $s_1^* = (C\ldots)$ (i.e. she starts by playing continue). Suppose this is a PSNE. Then by definition,

$$g_1(s_1^*, s_2^*) \geq g_1((S\ldots), s_2^*) = 1.$$

That is, the payoff that 1 expects to get by continuing most be at least as big as the payoff she gets by stopping at the first node, namely 1. But by the construction of the payoffs of the game, this means that $s_2^* = (C\ldots)$, since if 2 was staying stop right away, 1 would get payoff of 0. But for this to be a PSNE strategy for for 2, note that by the same logic 1 must be saying continue at his second node as well. We can continue this logic to show that both players must be playing C at every node, including the last. But at the last node, playing C is clearly not optimal; player 2 would want to deviate to playing S instead. Thus we have a contradiction, and so s^* where $s_1^* = (C\ldots)$ is not a PSNE strategy profile.